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# THE ARITHMETIC TEACHER
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# THE ARITHMETIC TEACHER

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# A Study of the Quantitative Values of Fifth and Sixth Grade Pupils*


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## Introduction

Observations made of children in the fifth and the sixth grades have revealed a weakness in pupil understanding of the quantitative concepts commonly used in these grades. A pupil may solve a verbal problem, report the results, and defend the algorism without really understanding the meaning of the answer. If, for example, his answer is sixteen feet, the child may be unable to demonstrate within reasonable tolerances the distance the results suggest. If the answer is $3\frac{1}{2}$ pounds, many pupils cannot differentiate tactually between this weight and some others within several pounds of it. Temperatures, although the pupils hear them announced daily on radio and television stations, may have relatively little meaning to fifth and sixth grade children. Likewise, time and capacity measures often appear to be only words to intermediate grade pupils.

## The Purpose of the Study

This research was designed to study some of the quantitative concepts of fifth and sixth grade pupils in five aspects of measurement. Measures were selected with which pupils of this age have frequent contacts: weight, linear distance, temperature, time, and liquid capacity. The purpose was to find how accurately fifth and sixth grade pupils can estimate and measure the following ten quantities:

1. The weight of a bar of plumber's lead ($4\frac{1}{2}$ pounds)
2. The weight of a blackboard eraser (2 ounces)
3. The weight of a block of wood (8 ounces)
4. The length of a piece of rope (16 feet)
5. The thickness of a lead pencil (5/16 inches)
6. The circumference of a basketball (30 inches)
7. The room temperature
8. The outdoor temperature
9. Time required for the sand to run through an egg timer (3 minutes)
10. The amount of water in a half-filled pail (6 quarts)

## Procedure

Pupils were interviewed individually outside the classroom. First they estimated each quantity as the teacher presented it. They could manipulate movable objects, use makeshift aids or devices, or assist themselves as they saw fit. After completing all of the estimates, each pupil selected from a table, on which were placed several different kinds of measuring devices, one he thought suitable for each measuring task. For the weights, a set of household dial scales and a set of spring-balance scales were provided. For linear distance, the work table contained a foot ruler, a yardstick, a six-foot jointed ruler and a fifty-foot tape measure.


* A research project supported by a research grant from the Central Research Fund.

For temperature measures an inside and an outside thermometer were conveniently placed. Wall clocks with sweep-second hands and watches with second hands were available for the time measurement. Standard metal quart and pint measures, glass quart milk bottles, and glass two-quart milk bottles were supplied for measuring the water. A second pail was provided for the disposal of the water.

Classroom teachers and student teachers in several schools scattered throughout Pennsylvania gathered the data used in the study. These teachers followed uniform instructions and used a common form for recording the data. Each teacher presented the same kinds of objects to be measured. Each provided comparable measuring tools for pupil use. The writer analyzed and summarized the data, which consisted of the recorded estimates, reports of measures, notes taken during interviews, and in some cases tape recording of pupils' comments.

## The Nature of the Data

The following classification was made of pupil estimates and reports of pupil measurements of the ten quantities used in the study:

1. The index and the per cent of estimated error.
2. The frequency of overestimate, underestimate, correct estimate, and of wrong estimate.
3. The index and per cent of measurement error.
4. The frequency of overmeasure, undermeasure, correct measure, and of wrong measure.
5. The index and per cent of estimate error in weight, linear distance, temperature, time, and liquid capacity.
6. The index and the per cent of measurement error in weight, linear distance, temperature, time, and liquid capacity.
7. The correlation between scores on an arithmetic achievement test and the percentage of estimate error.
8. The correlation between scores on an arithmetic achievement test and the percentage of measurement error.

The indexes of estimate error and of measurement error were determined by dividing the amount of error by the actual measurement of the quantity. For example, if the weight of a block of lead weighing $3\frac{1}{2}$ pounds was estimated to be $10\frac{1}{2}$ pounds, the error would be 7 pounds. That error divided by $3\frac{1}{2}$ would give a quotient of 2, the index of estimate error. If the pupil used an incorrect medium of measurement in making his estimate, or if he refused to attempt a guess of the quantity, no index could be computed. Likewise, a measurement attempted in the wrong terminology or a refusal to attempt a measurement did not rate an index figure. Per cents of measurement error and of estimate error were computed with 100% as the maximum error possible. A measurement or an estimate using wrong terminology or one not attempted was considered 100% wrong. Any index of error over 1.0 was also assumed to be 100% wrong. Wrong estimates or measures were those which showed incorrect terminology or procedure.

It was recognized that children of this age are not especially adept or experienced with measuring tools and sometimes not very well co-ordinated physically. Reasonable error was allowed in their application of the measures. The errors that were reported were those which in the opinion of the writer represented actual mistakes in judgment and in ability to read and manipulate the measuring devices.

A final source of information in the study was anecdotal in nature. This information included the interesting deviations from accepted patterns of measurement. A number of children, for example, weighed objects in inches; several used spring-balance scales for measuring temperature; and many measured the depth of the water with a ruler.

## The Pupils in the Study

Two classrooms in the Kane, Pennsylvania, Area Schools were selected for this study. One was a sixth grade and the other a combined fifth- and sixth-grade group. Two sixth grade classes from the College Area Schools, State College, Pennsylvania, were also chosen. A fifth grade class from Lock Haven, Pennsylvania, provided the fifth group of pupils for the investigation. There were, altogether, 108 sixth grade children, 55 boys and 53 girls. Fifth graders totaled 39, with 17 boys and 22 girls.

TABLE I

MEAN SCORES BY 108 SIXTH GRADE PUPILS AND BY 39 FIFTH GRADE PUPILS WHO PARTICIPATED IN THE STUDY OF QUANTITATIVE CONCEPTS ON THE CALIFORNIA TEST OF ARITHMETIC, FORM W; THE NUMBER ABOVE AND THE NUMBER BELOW GRADE NORMS

| Group | Number | Mean score | Norm | Number of pupils above the norm | Number of pupils below the norm |
|---|---|---|---|---|---|
| Sixth grade | 108 | 80.3 | 69 | 81 | 27 |
| Fifth grade | 39 | 55.3 | 49 | 29 | 10 |

All pupils in the study took the California Test of Arithmetic, Form W, for Elementary Grades 4, 5, and 6. Table I shows the results of this test. The mean score made by the sixth grade pupils was considerably higher than the norm for sixth graders at the time of the year the test was taken. The fifth graders likewise exceeded the grade norms. Approximately 75% of all of the fifth and sixth graders made scores above the norms for their respective grades.

The estimates by the pupils will be presented first, to be followed by the data concerned with measurement. The index both of estimate and of measurement error shows something of the latitude, the range of erroneous concept reported by the pupils. The percentage figures are somewhat more conservative, but they show the depth and persistence of the wrong understandings of measure.

Table II shows that sixth grade pupils missed the right estimate by an average of almost $1\frac{1}{2}$ times the real values. Fifth graders missed on their estimates by almost 6 times the actual measurements. Several boundless guesses (ranging up to 2000 pounds for the $4\frac{1}{2}$ pound block of lead) raised the averages and upset the statistical significance as well. The $t$ ratio between the two average indexes of error is 1.6, not significant. Boys, however, estimate more accurately than girls do when the index averages are the data considered.

TABLE II

MEAN INDEX OF ERROR OF ESTIMATE OF TEN QUANTITATIVE CONCEPTS BY 108 SIXTH GRADE PUPILS AND BY 39 FIFTH GRADE PUPILS

| Group | Mean index of estimate error | Mean index for boys | Mean index for girls |
|---|---|---|---|
| Sixth grade | 1.44 | .82 | 2.08 |
| Fifth grade | 5.79 | 2.49 | 8.24 |

$t$ ratio, sixth and fifth grade means 1.6 (not significant)
$t$ ratio, means of boys and of girls 2.2 (above 5% level of confidence)

TABLE III

MEAN PER CENT OF ERROR OF ESTIMATE OF TEN SELECTED QUANTITATIVE VALUES BY 108 SIXTH GRADE PUPILS AND BY 39 FIFTH GRADE PUPILS

| Group | Mean per cent of estimate error | Mean per cent for boys | Mean per cent for girls |
|---|---|---|---|
| Sixth grade | 45 | 43 | 51 |
| Fifth grade | 61 | 56 | 64 |

$t$ ratio, fifth and sixth grade norms 7.4 (.01 level of confidence)
$t$ ratio, means of boys and of girls 3.1 (.01 level of confidence)

TABLE IV

MEAN INDEX OF ERROR IN THE MEASUREMENT OF TEN SELECTED QUANTITIES BY 108 SIXTH GRADE PUPILS AND BY 39 FIFTH GRADE PUPILS

| Group | Mean index of measurement error | Mean index for boys | Mean index for girls |
|---|---|---|---|
| Sixth grade | .78 | .49 | 1.09 |
| Fifth grade | 2.39 | .63 | 3.74 |
| *t* ratio, fifth and sixth grade means 1.0 (not significant) | | | |
| *t* ratio, means of boys and of girls 1.3 (not significant) | | | |

TABLE V

MEAN PER CENT OF ERROR IN THE MEASUREMENT OF TEN SELECTED QUANTITIES BY 108 SIXTH GRADE PUPILS AND BY 39 FIFTH GRADE PUPILS

| Group | Mean per cent of measurement error | Mean per cent for boys | Mean per cent for girls |
|---|---|---|---|
| Sixth grade | 23 | 21 | 26 |
| Fifth grade | 35 | 35 | 35 |
| *t* ratio, fifth and sixth grade means 3.9 (.01 level of confidence) | | | |
| *t* ratio, means of boys and of girls 1.6 (not significant) | | | |

When the per cent of estimate error is considered, sixth grade pupils estimate significantly more closely than fifth graders. Boys are superior to girls in their ability to estimate quantitative values. These statistics are reported in Table III. The average percentage of estimate error for fifth grade pupils is over 60% actual value; that of sixth graders almost half of the actual measurements involved.

Gross errors in measurement are best shown through the index of measurement error statistics. When sixth graders used the measuring instruments, they made errors averaging more than three-fourths of the actual values. The fifth grade pupils made mistakes with measuring tools averaging almost two and one-half times the real values of the quantities they measured. Table IV presents this information in greater detail.

After the gross error in the index of measurement (Table IV) has been reduced to per cents not exceeding 100%, substantial evidence remains that intermediate grade pupils are unable to work effectively with common measuring tools. Sixth grade pupils, when allowed to measure the quantities, showed error amounting to almost one-fourth the quantities they sought to measure. Fifth graders made mistakes totaling over one-third of the real values. Little difference in this activity seemed to occur whether boys or girls did the measuring. Table V shows the percentages of measurement error.

A comparison of the per cents of overestimates, underestimates, right estimates, and wrong estimates is shown in Table VI. Sixth grade pupils were somewhat more conservative in their estimates than fifth graders. The sixth grade participants overestimated less often and underestimated more often than the fifth graders. There was no significant difference in the number of correct estimates. The number of wrong estimates was significantly higher among fifth graders than it was among the sixth grade pupils. In comparing the per cents for boys and girls, only in the frequency of right estimates did any significant differences occur. Boys made a significantly higher per cent of right estimates than girls. This fact supports data in preceding tables which indicated that boys are more effective in estimating quantitative values than are girls.

When sixth grade pupils measured each of the ten quantities, they appeared to exhibit somewhat the same characteristics as they did in their estimates. Although the per cent of overmeasures by sixth graders was not significantly lower than fifth graders, the per cent of undermeasures was significantly higher. Right measures occurred more frequently and wrong measures less often among sixth graders than among fifth graders. These percentage statistics support the previous measurement statistics and show no significant differences between boys and girls in the frequency of each category of measurement. Table VII shows the per cents of overmeasure, undermeasure, right measure, and wrong measure for the fifth and sixth grade pupils.

Tables II through VII have presented the the basic data on estimates, measurements, and the nature of the errors made by the pupils. These data indicate both the lateral spread and the depth of quantitative error. The index of estimate error and the index of

TABLE VI

MEAN PER CENTS OF OVERESTIMATES, UNDERESTIMATES, RIGHT ESTIMATES, AND WRONG ESTIMATES BY 108 SIXTH GRADE PUPILS AND BY 39 FIFTH GRADE PUPILS IN ESTIMATING TEN QUANTITATIVE VALUES

| Per cents of | Sixth grade pupils | Fifth grade pupils | Sixth grade boys | Fifth grade boys | Sixth grade girls | Fifth grade girls |
|---|---|---|---|---|---|---|
| Overestimates | 37 | 42 | 36 | 40 | 39 | 43 |
| Underestimates | 46 | 37 | 46 | 40 | 46 | 36 |
| Right estimates | 13 | 11 | 14 | 13 | 11 | 9 |
| Wrong estimates | 4 | 10 | 4 | 7 | 4 | 12 |

*t* ratio, means of fifth and sixth grade overestimates 2.1 (.05 level of confidence)
*t* ratio, means of fifth and sixth grade underestimates 4.4 (.01 level of confidence)
*t* ratio, means of fifth and sixth grade right estimates 1.1 (not significant)
*t* ratio, means of fifth and sixth grade wrong estimates 3.8 (.01 level of confidence)
*t* ratio, means of boys' and girls' overestimates .9 (not significant)
*t* ratio, means of boys' and girls' right estimates 3.6 (.01 level of confidence)
*t* ratio, means of boys' and girls' wrong estimates 1.0 (not significant)

TABLE VII

MEAN PER CENTS OF OVERMEASURES, UNDERMEASURES, RIGHT MEASURES, AND WRONG MEASURES BY 108 SIXTH GRADE PUPILS AND BY 39 FIFTH GRADE PUPILS IN THE MEASUREMENT OF TEN SELECTED QUANTITATIVE VALUES

| Per cents of | Sixth grade pupils | Fifth grade pupils | Sixth grade boys | Fifth grade boys | Sixth grade girls | Fifth grade girls |
|---|---|---|---|---|---|---|
| Overmeasures | 18 | 22 | 17 | 24 | 19 | 21 |
| Undermeasures | 24 | 30 | 23 | 31 | 24 | 29 |
| Right measures | 51 | 34 | 54 | 35 | 49 | 33 |
| Wrong measures | 7 | 14 | 6 | 10 | 8 | 17 |

*t* ratio, means of fifth and sixth grade overmeasures 1.8 (not significant)
*t* ratio, means of fifth and sixth grade undermeasures 2.5 (.02 level of confidence)
*t* ratio, means of fifth and sixth grade right measures 5.5 (.01 level of confidence)
*t* ratio, means of fifth and sixth grade wrong measures 2.8 (.01 level of confidence)
*t* ratio, means of boys' and girls' overmeasures 1.2 (not significant)
*t* ratio, means of boys' and girls' undermeasures 0.4 (not significant)
*t* ratio, means of boys' and girls' right measures 1.4 (not significant)
*t* ratio, means of boys' and girls' wrong measures 1.1 (not significant)

TABLE VIII

MEAN INDEXES OF ESTIMATE ERROR AND OF MEASUREMENT ERROR BY 108 SIXTH GRADE PUPILS AND BY 39 FIFTH GRADE PUPILS OCCURRING IN EACH OF THE FIVE CATEGORIES OF MEASURE: WEIGHT, LINEAR DISTANCE, TEMPERATURE, TIME, AND CAPACITY

| | Index of estimate error | | | | | Index of measurement error | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| *Group* | *Weight* | *Lin. meas.* | *Temp.* | *Time* | *Cap.* | *Weight* | *Lin. meas.* | *Temp.* | *Time* | *Cap.* |
| Sixth grade | 3.69 | .56 | .18 | .40 | .32 | .68 | 1.15 | .04 | .84 | .17 |
| Fifth grade | 14.57 | .96 | .30 | 3.28 | .65 | 8.12 | 1.22 | .04 | .71 | .16 |

TABLE IX

MEAN PER CENTS OF ESTIMATE ERROR AND OF MEASUREMENT ERROR BY 108 SIXTH GRADE PUPILS AND BY 39 FIFTH GRADE PUPILS IN EACH OF THE FIVE CATEGORIES OF MEASURE: WEIGHT, LINEAR DISTANCE, TEMPERATURE, TIME, AND LIQUID CAPACITY

| | Per cent of estimate error | | | | | Per cent of measurement error | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| *Group* | *Weight* | *Lin. meas.* | *Temp.* | *Time* | *Cap.* | *Weight* | *Lin. meas.* | *Temp.* | *Time* | *Cap.* |
| Sixth grade | 71 | 40 | 18 | 31 | 66 | 24 | 27 | 7 | 20 | 46 |
| Fifth grade | 84 | 52 | 33 | 54 | 84 | 46 | 41 | 13 | 43 | 49 |

measurement error reveal something of the breadth of incorrect responses. The per cent of estimate error and the per cent of measurement error speak of the persistence of pupil difficulty with quantitative values. It seems a bit incredible to believe that sixth grade pupils err in their estimates by almost one and one-half times the actual quantity and fifth graders by nearly six times the actual quantity. It is equally surprising to learn that, given the tools, sixth grade pupils measured accurately in only about half of the opportunities given them. Fifth graders were even less effective, measuring correctly in slightly over one-third of the opportunities.

For all pupils, the highest index of estimate error occurred in the weights. Estimates of linear measure were somewhat more accurate, and temperature estimates were probably the most nearly correct. Table VIII presents the index of estimate error and the index of measurement error for each of the five categories of measurement.

Although sixth grade pupils were able to weigh objects more accurately than they could estimate them, fifth graders seemed to do little better with the scales than with their estimates. Fifth grade pupils averaged nearly eight times the weight value in their index of measurement error when they used the scales. Linear estimates were somewhat more accurate for both sixth and fifth graders than were their measures.

Per cents of estimate and of measurement error show that the greatest distortion in concept appeared in weights and the lowest in temperature. Both in estimating and in measuring, liquid measures appeared to give the pupils of both grades considerable difficulty. This was apparent in the number of pupils who failed to find the appropriate medium for measuring the water in the bucket. Almost one third of the pupils chose an instrument to measure linear distance and measured the depth of the water. Table IX shows the mean per cent of estimate error and the mean per cent of measurement error in each of the five categories.

## Relationship to Pupil Achievement

The scores on the California Arithmetic Test, Form W, for Grades 4, 5, and 6 were correlated with the per cent of estimate error and the per cent of measurement error for each pupil. The Pearson product moment coefficient of correlation was used. The correlation between the arithmetic achievement test and the percentage of estimate error was .379. The correlation between the achievement test scores and the percentage of measurement error was .547. It was somewhat surprising to find the correlation between measured arithmetic achievement and ability to estimate quantitative values as low as it was. Teachers often assume that ability to achieve satisfactorily on an arithmetic achievement test is an accurate measure of all positive arithmetical concepts. The teacher might have expected the correlation with measurement ability to be higher if she assumed that achievement test scores reflect a well-rounded arithmetical development.

## Anecdotal Report

Certain pupil errors, while they probably have little statistical importance, persisted throughout the study. They are of anecdotal interest, and some of them are worth reporting. Seven pupils referred to the scales as "a weigher." Nine did not know the meaning of "thickness" when the teacher spoke of the "thickness of the pencil." Twenty-two pupils wrapped the tape measure around the lead pencil to determine its thickness. Four were unable to measure the distance around the basketball. Ten fifth and sixth graders used a ruler or a yardstick instead of a tape measure to measure the circumference of the ball. Eight pupils insisted that the blackboard eraser "didn't weigh anything." Seven children used spring-balance scales to measure temperature, and seven others used a barometer which was unintentionally left in the room. Six pupils weighed in inches, and another six could not read the scales. Forty-eight pupils used a yardstick or a ruler to measure the depth of the water in the bucket, rather than its liquid capacity.

On the positive side, there were numerous references to social understanding of the measuring devices. Many pupils recognized the tools and reported having used them at home or in some other social situation. A large number chose the tape measure to find the circumference of the ball because "this is the only one we can wrap around the ball." The egg timer was familiar to many children who said "three minutes" without hesitation. Pupils appeared to be concerned about what they were doing: some were highly amused at the enormity of their estimate error, and many showed real interest in the measurement tools. Teachers reported some surprise and embarrassment over the ineffectiveness of many of their pupils in the use of measurement devices.

## Conclusions

As the result of this investigation certain conclusions have been reached which appear to be adequately supported by the data.

1. The pupils studied during the investigation were typical of fifth and sixth grade pupils throughout the Pennsylvania area. The schools were widely scattered, the average arithmetic achievement was at or above the norm for the grade the pupils were in, and no preparation for the investigation had been made, so far as the pupils were concerned.
2. Sixth grade pupils were more effective in estimating quantities than were fifth graders. This suggests that estimating can be learned during a school year, and that pupils grow in this skill.
3. Boys estimate more accurately than girls. Although there is no evidence to prove it, boys appeared to be more familiar with measurements, with measuring devices, and with the approximate values each quantity represented.
4. Sixth graders showed a 45% error in estimating. Fifth grade pupils averaged 61% in their estimate error. The average per cent of estimate error was somewhat over half for all of the pupils in the study.
5. Errors in measurement by sixth graders averaged .78 of the actual values; by fifth graders the error was 2.39 times the real values. These statistics show the gross error and something of the range of mistakes in measurement.
6. There was no significant difference between boys and girls in the index of measuring error. Apparently boys and girls used the tools of measurement with almost equal effectiveness.

7. Sixth grade pupils averaged almost one fourth (23%) measurement error when they used the measuring tools. Fifth graders averaged over one third (35%) measurement error in their efforts with the instruments.

8. There was no significant difference between boys and girls in the per cent of measurement error.

9. Sixth grade pupils overestimated less frequently than fifth graders.

10. Sixth grade pupils underestimated more frequently than fifth graders.

11. There was no significant difference between fifth and sixth grade pupils in the number of right estimates.

12. Sixth grade pupils had fewer wrong estimates than fifth graders.

13. Boys exceeded girls in the number of right estimates.

14. Sixth grade pupils undermeasured more often than fifth graders, were correct in their measures more often, and were wrong less often than fifth graders.

15. There was no statistical difference in the frequency of overmeasure, undermeasure, right measure, and wrong measure when boys and girls were considered as dichotomous variables.

16. The greatest discrepancy in measuring and in estimating occurred in weights. The smallest discrepancy in both of these activities was found in temperature.

17. Pupils appeared to be more accurate in estimating linear distance than in measuring it.

18. The correlation between the test scores on the California Test of Arithmetic and the per cent of measurement error was .547.

## Implications of the Study

Previous studies of problem-solving behavior among fifth and sixth grade pupils have revealed many instances of erroneous and incomplete quantitative concepts. There is reason to believe that such faulty information about quantitative matters affects verbal problem solving in arithmetic. This study was undertaken to determine the nature and extent of the functional knowledge of measures among upper elementary grade pupils. It is not easy to say just how much a given boy or girl ought to understand about measures. There are no norms or standards which establish for teachers guaranteed quantitative competence. Even adults have no recognized boundaries for fluctuating quantitative skills.

This study does not attempt to establish norms for pupil competence in measures. It presents to teachers, through an analysis of pupil error, something of the status of fifth and sixth grade proficiency with measures. It offers probable reasons for ineffective, vague analysis of verbal problems which deal with quantity. It indicates that measures are learned, that functional knowledge about them may be among the criteria for intellectual growth and maturity. It reveals that limited use has been made of objective devices for teaching the practical applications of measures. It suggests lack of pupil identification with the factual material commonly taught about measurements.

Finally, the relatively low coefficients of correlation between quantitative concepts and achievement test scores place further limits upon the use of these scores to measure total arithmetic accomplishment.

EDITOR'S NOTE. Whether the particular indices used in this study are the best possible or whether they will be adopted by other research workers remains to be seen. However, the evidence is sufficiently clear to make a strong indictment against our present elementary program in this quarter.

The enormity of pupils' errors, both in estimating and in actual measurement, when dealing with some of the most common quantitative concepts upon which our textbook problems are based, certainly accounts for some of the pupils' troubles in problem solving, but leaves us wondering how we are spending our time.

This might well be an area in which research upon content and method, as well as status, should be promoted.

# Depth Learning in Arithmetic—What Is It?[1]

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The acquisition of skills has been consistently recognized as one of the basic aims of education to which the elementary school should contribute. From colonial times to the present, much of the attention of schools and of teachers has been centered on this aspect of education. Furthermore, this preoccupation with the development of skills has been associated largely with the "three R's." Reading, writing, and arithmetic, frequently referred to as tool subjects, have been presumed to encompass all the skills which are of real importance in the education of children.

Although it is true that effective use of language and numbers does require high-level skills of a complex sort, the three R's involve far more than skills. One of the most damaging and limiting conceptions in elementary education arises from the point of view that developing competence in language and arithmetic is largely a matter of gaining command of specific skills. Curriculums and teaching procedures based on this idea result in rote learning, the memorization of specific facts, and the teaching of "things."

Learning, in its proper sense, is concerned not with "things," but with the "meaning of things." All forms of language and mathematics are essentially concerned with meanings. They represent ways of gaining, interpreting, and transmitting ideas, and explaining relationships. These processes rest for effectiveness on basic skill, but the controlling principle in learning is meaning and relationship.

Efficiency in habit and skill alone is not a satisfying end-point of education, if for no other reason than that life itself is not a simple problem in logistics, a problem for which any good Answer Book has the precise formula worked out to the fourth decimal place!

When learning is at its best, it will provide opportunities for the child to produce original ideas, to be active in his own learning, and to give range to his imagination. Somewhere in our teaching procedure we must provide time for contemplation, for experimentation, and for wondering. Was it not Einstein who said to us, "Those who have ceased to wonder are as good as dead"? Is it possible that in our own failure to wonder we do not see the importance of opportunities for children to wonder? All too often in the pressure for efficiency, in our desire to make the curriculum more "rigorous," and in the crowding of assignments up to the pupil's capacity, we forget to save time for the learner to give free expression to the understandings he is developing: to question, to wonder, and to experiment beyond accepted patterns of operation in arithmetic. Like automatons, children are trained to record, retain, and reproduce larger and larger amounts of material to which someone else has given shape and form. Seldom do they experience the excitement of discovery, never the risk of being wrong, never a moment in which they can say, "Poor though it is, it is my own!" For these children learning in arithmetic has taken on a routine quality. It is a surface thing lacking in meaning and in depth.

It was just a few short years ago that arithmetic was the "red-haired stepchild" in the elementary curriculum. Now we ride the crest—due to public interest and concern as well as interest and concern within the pro-

[1] Presented at the 38th Annual Meeting of The National Council of Teachers of Mathematics, Buffalo, New York, April 22, 1960.

fession. For the first time we have the financial support to bring the changes in the curriculum which are necessary to meet the needs of our times and the capabilities of our students. In our headlong pursuit of excellence we *must not* lose sight of a very basic goal in teaching—the development of the ability to think quantitatively. It would be very easy under present pressures to make *record*, *retain*, and *reproduce* the three R's of arithmetic!

This is not to say that acquisition of the basic facts and principles of arithmetic is unnecessary and unimportant. It is to say that if real learning is to take place on the part of the student, then one must view knowledge as a means, not as an end, in arithmetic education. The building of accurate, well-organized concepts is, from the point of view of the learner, a creative process in which thinking plays a leading role. All thinking is dependent upon a well-organized body of knowledge, thoroughly understood and pertinent to the problem. As we all know, "Wisdom does not come to those who gape at nature with an empty head."[2]

## Developing Depth in Learning

Depth learning in arithmetic is not developed, then, by emphasis upon skills alone. Neither is it developed solely by providing children with "enrichment" activities or with content of a more difficult nature taught at an advanced level. It is developed in quite a different way, that is:

1. by confronting the student with challenging problems—but problems within his power of comprehension,
2. by leading him, from the very beginning, to see the futility of thought without dependable data,
3. by maturing him in those methods of disciplined thought that have been found to facilitate the work of mathematicians,
4. by providing opportunities to discover and to find original solutions, and
5. by steadily encouraging him to new levels of creative thinking.

Depth learning in arithmetic results from a way of teaching which encourages reflective thinking, which permits experimentation and originality, and which holds steadily before the learner the *need for knowing* the facts and principles necessary for clear thought. Given these things, children will move on to concepts of greater difficulty and the development of greater skill in handling quantitative relationships.

Note that I have not spoken here of abstraction, but I have spoken of precision in thinking and meaning. Thought may be precise whether abstract or concrete. Note that I have not spoken of rigor. Concepts correctly understood and precisely expressed at any level will embody a standard of rigor appropriate to the thinker's maturity, and as he grows in knowledge and skill and in his ability to sense relationships, his need for higher standards also will grow.

Not all children in the elementary school are equally able. They do not all have the same ability, and as we try to provide opportunities for them to add depth to their learning, it appears that we may attempt the impossible.

Ambitious goals that we set up will be attained fully by some, to a modest degree by many, and to a limited degree by others. Nevertheless, *it is sound policy* to encourage all children to think as hard as they can and to understand as much as they can. It is better to see through a glass darkly than not to see at all. The opportunity to attain adequate understanding and to add depth to their learning should be open to all children.

The unquestionable evidence of differences in learning ability must be faced squarely by those of us responsible for instruction in arithmetic. This need not lead to a pessimistic or defeatist point of view, but to the establishment of attainable goals and the devising of more effective methods for their accomplishment.

Today, there appears to be common agreement among teachers of arithmetic that there should be little, if any, differentiation of topics in the elementary school to meet individual needs. The differentiation, when made, should be in terms of *depth* and scope of learning. As Roland Smith states,

[2] Morris R. Cohen, *Reason and Nature* (New York: Harcourt Brace & Company, 1931), p. 17.

"... the amount of concrete background in any topic can be varied. Rates of learning can be varied. The extent of any topic can be varied, and each child can be expected to do work only up to his own capacity to learn."[3]

Providing for differentiation in scope and in depth is made easier by a method of teaching which leads children to think as a mathematician would think. It enables them to solve problems that are not exactly like those solved in the classroom or in the textbook. Furthermore, it enables them to deal precisely with precise ideas, and to work with symbols as an aid to thought, full of meaning and conveying ideas which can be understood.

### Selected Experiences

Children are particularly delighted with learning carried on in this way. Recently I watched children in a second grade, who had developed some facility with basic addition and subtraction facts, work through a series of questions in which they were solving for the "missing number." For example:

$$5+\square=7 \qquad \triangle+9=10$$
$$2+\square=8 \qquad \triangle+6=13$$

Each child was encouraged to tell "how he thought" in finding the missing numbers. Then the children were asked to work independently to find pairs of numbers which would make these equations true:

$$\triangle+\square=7 \qquad \square+\triangle=5$$

Class evaluation revealed that several different number pairs could be used in each example. Some of the children who experienced no difficulty here were encouraged to go on to the slightly more difficult situation of:

A

$$6+3+\square=10$$
$$4+\triangle+3=9$$
$$\square+2+4=8$$

B

$$6+2+\square=10$$
$$4+2+\triangle=9$$
$$1+\triangle+4=8$$

The children were encouraged to create similar exercises with which to challenge their classmates.

At a fifth or sixth grade level, I watched children, who have had opportunities to do this type of work, extend their thinking with exercises in which they decided whether a number sentence was *true* or *false* depending on the numbers used. For example:

A. $3+5=8$ (true)
   $2+7=8$ (false)

B. Is the number sentence, $A+4=9$, true?
- if $A=4$
- if $A=5$
- if $A=6$

C. Is this sentence, $A+B+6=12$, true or false?
- if $A=3$ and $B=4$
- if $A=2$ and $B=4$
- if $A=2$ and $B=3$
- if $A=2\frac{1}{2}$ and $B=3\frac{1}{2}$
- if $A=5.2$ and $B=.8$

The children were encouraged to find other pairs of numbers which made the sentence true and to share their thinking with the whole class.

Also, I observed a sixth grade teacher working with a group of children of varying abilities. The children worked as a group to solve the following:

- If $T=4$; $R=6$, and $V=5$, find the number which will make each of the following number sentences true.
- $R+T+V=?$
- $T\times V\times R=?$
- $(R-T)+V=?$
- $T+R\div V=?$

After solving several exercises like these, the children then developed other similiar exercises to share with the class.

As children develop understandings and competence with the basic multiplication facts, they may be given opportunities to stretch their thinking on exercises like this:

$$\begin{aligned} 2\times 9 &= (2\times 4)+(2\times 5)\\ 5\times 7 &= (5\times 2)+(5\times 5)\\ &= (5\times 3)+(5\times 4)\\ &= (5\times 1)+(5\times 6)\end{aligned}$$

[3] Roland Smith, "Provisions for Individual Differences," *Twenty-first Yearbook* (Washington, D.C.: NCTM, 1953) p. 273.

**Can you complete the following by finding other ways to express the meaning of $4\times6$, $5\times8$, and $6\times7$?**

**$4\times6=$_____ $5\times8=$_____ $6\times7=$_____**

After instruction on how to multiply without pencil and paper pupils in one fourth grade were using six different ways of arriving at the product of $12\times25$:

1. $$\begin{array}{r} 10\times25=250 \\ 2\times25=\underline{\ 50} \\ 12\times25=300 \end{array}$$

2. $$\begin{array}{r} 4\times25=100 \\ 4\times25=100 \\ 4\times25=\underline{100} \\ \text{So } 12\times25=300 \end{array}$$

3. $$\begin{array}{r} 12=4\times3 \\ 3\times25=\ 75 \\ 4\times75=300 \\ \text{So } 12\times25=300 \end{array}$$

4. $$\begin{array}{r} 12=3\times4 \\ 4\times25=100 \\ 3\times100=300 \\ \text{So } 12\times25=300 \end{array}$$

5. $$\begin{array}{r} 25 \\ \underline{12} \\ 50 \\ \underline{250} \\ 300 \end{array}$$

6. $$\begin{array}{r} 12=2\times6 \\ 6\times25=150 \\ 2\times150=300 \\ \text{So } 12\times25=300 \end{array}$$

Children whose written work indicates that they require little additional practice in multiplication and division profit from exercises which develop relational thinking. For example:

Having children tell how they think when finding trial quotients in division is not only helpful to individual children, but it helps all children in the class deepen their thinking. Recently, in a fifth grade, I watched children share their solutions to the following examples.

| | |
|---|---|
| $29/\overline{89}$ | BOB: The answer can't be as much as 4, since 29 is almost 30 and $4\times30=120$. The answer will be a little more than 3. |
| $51/\overline{426}$ | SUE: There are 2 50's in 100. In 400 there will be at least 4 times as many. I'll try 8 as a quotient figure. |
| $45/\overline{280}$ | ANN: There are 2 45's in 100. In 300 there will be about 6. I'll try 6 as a quotient figure. |

Occasionally current happenings can be used to bring depth into children's learning. When Pioneer IV was successfully placed in orbit, a fifth grade child related to his classmates that the *velocity* of the rocket was a little less than 25,000 mph at time of launching, that at $X+1$ the *speed* was about 11,800 mph, and at $X+2$ the speed was reduced to about 7,800 mph. Class discussion clarified for the children the difference between velocity and speed. The children noted that the rocket's speed at the end of the first hour was about half that at time of launching, and that at the end of the second hour the speed was reduced to about one-third the original speed. A speed of 25,000 mph had little meaning for the children until a little computation indicated that this was about 400 miles per minute. Knowing that it is about 400 miles across Michigan from north to south, the children began to gain an appreci-

***Tell how you think to get the answer***

| | | (a) | (b) | (c) | (d) |
|---|---|---|---|---|---|
| 1. | $\boxed{\begin{array}{r} 32 \\ \times16 \\ \hline 512 \end{array}}$ | $\begin{array}{r} 32 \\ \times\ 8 \\ \hline \end{array}$ | $\begin{array}{r} 32 \\ \times32 \\ \hline \end{array}$ | $\begin{array}{r} 64 \\ \times16 \\ \hline \end{array}$ | $\begin{array}{r} 16 \\ \times32 \\ \hline \end{array}$ |
| 2. | $\boxed{\begin{array}{r} 12 \\ 18/\overline{216} \end{array}}$ | $12/\overline{216}$ | $36/\overline{216}$ | $18/\overline{432}$ | $\begin{array}{r} 6 \\ 18/\overline{?} \end{array}$ |

ation of the rocket's speed when they realized it would flash across Michigan in one minute.

Another computation indicated that the rocket traveled more than 6 miles a second and, as one child said, "Oh, we would travel from Detroit to Ann Arbor in about 7 seconds!"

When children have gained an understanding of the decimal number system, scientific notation may be introduced in grade six as a shorter and faster way of working with large numbers. For example, the children have learned that in the decimal number system:

1. $10=10\times1 \longrightarrow 10=10^1$ (10 used as a factor once)
   $100=10\times10 \longrightarrow 100=10^2$ (10 used as a factor twice)
   $1{,}000=10\times10\times10 \longrightarrow 1{,}000=10^3$ (10 used as a factor 3 times)
   $10{,}000=10\times10\times10\times10 \longrightarrow 10{,}000=10^4$ (10 used as a factor 4 times).

They can extend this to include:

2. $10\times 100=\ ?$ or $10\times10^2=\ ?$

3. $100\times1000=\ ?$ or $10^2\times10^3=\ ?$

From exercises like these children can be led to discover that, when multiplying powers of the same base, they can add the exponents.

1. $10^3\times10^2=10\times10\times10\times10\times10=10^5=100{,}000$

2. $10^2\times10^2=10^{2+2}=10^4=10{,}000$

3. $2^3\times2^1=2^4=2\times2\times2\times2=16$

If the teacher thinks it advisable, the use of a dot to indicate multiplication may be introduced at this time, too, and children may be encouraged to write $2\times5$ as $2\cdot5$; $6\times7$ as $6\cdot7$, etc.

In helping children extend their understanding of the use of formulas to express relationships, a teacher of eleven-year-olds placed the following number series on the chalkboard and asked if they could find a quick way to find the sum of all numbers from 1 to 101 without adding:

a. $1\ 2\ \underline{\underline{3}}\ 4\ \underline{\underline{5}}=15$

b. $1\ 2\ 3\ 4\ \underline{\underline{5}}\ 6\ 7\ 8\ \underline{\underline{9}}=45$

c. $1\ 2\ 3\ 4\ 5\ \underline{\underline{6}}\ 7\ 8\ 9\ 10\ \underline{\underline{11}}=?$

d. $1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13=?$

After a little study the children said that the sum in each series was equal to the middle number times the last number in the series. This they expressed as "Sum $=M\times L$." Applying this formula to the original question, they found the sum of all numbers from 1 to 101 to be $51\times101$ or 5151.

Then the teacher called attention to the fact that the number series with which they had been working all ended with odd numbers. She asked, "Will the formula work with a number series ending with even numbers?"

$1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6=\underline{21}$

$1\ \ 2\ \ 3\ \ 4=\underline{10}$

$1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10=\underline{55}$

It was necessary to develop new formula for these series and the children developed three different ones:

1. (First+Last)×(½ of the Last)
   $(F+L)\times(\frac{1}{2}\times L)$
2. (Last+1)×(½ of the Last)
   $(L+1)\times(\frac{1}{2}\times L)$
3. (Last×Middle) or $L\times M$.

In this latter case the children worked with fractions and in the series "1 2 3 4 5 6" used the formula $L\times M=6\times3\frac{1}{2}=21$.

Recently I observed a fourth grade group working with number sequences. They had started with very simple sequences such as:

2, 4, 6, 8

7, 9, 13, 21

and were asked to write the next four numbers, then they advanced to irregular patterns such as 12, 14, 15, 17 and continued to write "the next four numbers." I watched a small girl place the following sequence on the blackboard and ask her classmates to complete it:

32 30 26 20 22 __ __ __ __

It was this same class who, at an earlier date, had worked successfully with numbers to the base of 7!

## In Conclusion

These are but a few illustrations of learning experiences used to add *depth* in meaning for children in arithmetic. The rule which we follow in achieving this kind of learning is a simple one:

*Broaden the context and lift the skill to the conceptual level,* by teaching so that children understand relationships, extend them to solutions of new problems, and have time to think, to question, to wonder.

To do this the teacher must be concerned with the development of the individual student. Each student builds his own ideas, and it is perfectly clear that in so doing his attempts will be attended by trial and error. It is equally clear that there is no substitute for the teacher either in stimulating the child's thinking or correcting and perfecting the ideas he forms. No administrative device, no amount of instructional equipment, important as it is, can take the teacher's place. Because of the complex nature of learning in mathematics, it is crucially important that teachers have adequate preparation. We, too, must add depth to our learning!

EDITOR'S NOTE. Professor Junge's own summary might well be extended as an editorial comment on her article. She comments that nothing can take the teacher's place. We might add that the printed development of material to extend the thinking of children, good as it is, pales by contrast with the original presentation. When the teacher's personality, side comments, and reaction to the audience are a part of the presentation, the effect is stronger still. If the reader will use these or similar ideas and develop for himself more and more expert presentations, Dr. Junge's purpose will have been accomplished.

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## What Is A Teacher?

"The teacher is a composite. A teacher must have the energy of a harnessed volcano, the efficiency of an adding machine, the memory of an elephant, the understanding of a psychiatrist, the wisdom of Solomon, the tenacity of a spider, the patience of a turtle trying to cross the freeway in rush-hour traffic, the decisiveness of a general, the diplomacy of an ambassador, and the financial acumen of a Wall Street wizard. She must remember always that she teaches by word but mostly by precept and example."—*Taken from "What Is A Teacher? by Jane C. Butler in* New York State Education, Vol. 44, p. 592.

# Arithmetic Concepts Possessed by the Preschool Child

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THE FIRST HALF of the twentieth century has seen many studies made by educators to determine the most desirable time to start a planned sequential instructional program in arithmetic for the young child. These studies have presented findings that vary from initial instruction starting in the first grade to a recommendation that beginning formalized instruction be delayed until the seventh grade. However, these studies indicate that, in general, the young child benefits when systematic instruction starts during his first-grade experiences.

It is interesting to note that previous studies of arithmetic and the young child were carried out after the child was well into or past his kindergarten year of public school. While formalized instruction in arithmetic is rarely started in kindergarten, an informal, incidental program in elementary number concepts is commonly taught. This leaves educators with unanswered questions regarding the extent of number knowledges and skills possessed by the preschool child at the start of kindergarten.

What do we know about five-year-olds? Actually, there is much to learn about these children as they come to us filled with enthusiasm about embarking on the great new experience of going to school. As public educators, we receive the child after he has had approximately five years of training and experience in the home. We accept the general fact that homes offer a great variety of backgrounds and meaningful experiences. We even go so far as to accept the wide differences among homes. We look at the five-year-old and say he is different from others in his peer group because of his family inheritance, his environment, number of siblings and their age relative to him, and the type of home life, travel, and personal experiences provided. We will go further and accept differences caused by a large number of less evident home experiences as well as by the partially understood pattern of child growth and development.

Accepting these differences, we have a tendency to overlook the specific differences in specific areas of knowledge. Are we guilty of taking the kindergarten child into our schools with only a vague notion of the number concepts that his previous five years have permitted him to acquire? In an attempt to answer this question, a study was made of the arithmetic concepts possessed by one hundred beginning kindergarten children in the Livonia, Michigan, public schools in the fall of 1957. The study was repeated again in February, 1960, with twenty-seven pupils, an entire midyear class of beginning kindergarten children, in the demonstration school on the campus of San Francisco State College in San Francisco, California. The children in the demonstration school are not a select group. Rather, they are all children of this age residing in an assigned attendance area determined by the San Francisco Unified School District.

The purpose of the study was to discover some of the specific number concepts possessed by the preschool child at the time of kindergarten entrance. In no way, whatsoever, has the study been conducted to permit interpretation as an inventory of a child's readiness for formal instruction in

arithmetic. Instead, it is hoped that the findings may bring about a closer scrutiny of what is generally accepted by teachers as normal number knowledge at the beginning of the kindergarten year, with the result that the planned instruction may be based realistically on the knowledges and needs of the five-year-old.

Two tests were developed to aid in securing the needed data. The first test was an individual oral interview containing fifty-seven responses. Number knowledges tested included abstract counting by ones and tens, rational counting by ones and twos, number sequence, ordinal numbers, identification of number symbols, recognition of number quantities, ability to name and recognize common instruments of measurement, recognition of the coins and dollar bill in the United States monetary system, and the ability to combine smaller coins to total the value of larger coins and the dollar bill.

The second test constructed was a written picture-test containing forty-five responses. Arithmetical concepts that could be presented clearly in picture form were selected. These included premeasurement understandings, such as *largest*, *smallest*, *tallest*, *shortest*, *longest*, *under*, *inside*, *beside*, *nearest*, and *farthest;* recognition of simple geometric figures, such as *circle* and *square;* telling time to the full and half-hour; counting less than ten items and recognizing the written symbol for the total; reading the number words to ten and recognizing their written symbols; fractional parts of a whole, such as *one-half*, *one-third*, and *one-fourth;* fractional part of a group of items, such as *one-half;* simple addition combinations in oral problem situations; simple subtraction combinations in oral problems with the ability to recognize the result in the written symbol. Considerable effort was made to see that each item and picture were ones familiar to five-year-olds.

The data secured from both groups of children who were presented with the tests were strikingly similar. In generalizing the findings, both groups will be treated as one.

Considerable variance in ability to do rote counting by *one* was evidenced, but every child displayed some facility in this number knowledge. The mean for the entire group was approximately nineteen, while five children were able to count to one hundred or higher. The tabulation showed that four of these five children were boys, with one boy reaching the total of 112.

The majority of the children were unable to do rote counting by ten, but approximately 25 per cent exhibited some ability to perform this arithmetical skill. A few demonstrated their ability to do rote counting by ten to one hundred. Some individual comments made during this portion of the test were, "I thought there was only one way to count!" while another child promptly counted to sixty by fives.

As might be expected, there was a marked similarity between the ability of the children to do rational counting by one and rote counting by one. The mean, again for the entire group, was about nineteen. For some, as the higher numbers were reached, there was slight confusion of one-to-one correspondence. A few counted in blocks of sequence, jumping several decades and continuing without error.

Few children displayed an understanding of number sequences other than the sequence of numbers by one. Less than 10 per cent understood a sequence of odd numbers, but approximately 20 per cent succeeded when a sequence of even numbers less than ten was used. Only about 5 per cent were able to respond with an understanding of a number sequence by five when the total was no higher than the number twenty. Most of the incorrect responses given in the variety of sequences consisted of giving the next digit (e.g., 3, 5, 7, *8*).

The results of the items concerned with ordinal numbers showed that more than half of the children had some understanding of this concept. Approximately 95 per cent understood the ordinal number *first*, while slightly more than 70 per cent knew *middle* and *last*. At least 50 per cent displayed a knowledge of the ordinal numbers *second* and *fourth*.

These beginning kindergarten children possessed considerable understanding of number selection skills and were able to recognize a quantity of items numbering less than four immediately. Some were able to recognize more than four items, but less than nine items. All of the children were able to select quantities of three or less. The majority of the children resorted to counting the objects one at a time, but a few were able to group items for quicker and more efficient recognition.

In situations where flash cards with a variety of pictured items were used, some rather enlightening ability to estimate was evidenced. Ninety-three per cent of the children immediately recognized two items when flashed, but this percentage dropped quite drastically after four items had been flashed. When eight items (the maximum number used) was flashed, the percentage of children responding accurately was 21 per cent. More than one-fourth of the children estimated by regrouping in the higher numbered items and then asked if they could check their answers by counting. Some actual responses were: "Let's see, two and two more, and two—that's six of them"; "Two and two, four. Two fours—there's eight." When asked to point out the picture containing the number of candles he would have on his birthday cake to show how old he was, one boy declared, "I'm 4½," and he immediately picked out the cards picturing four and five candles.

It was evident that a large percentage of beginning kindergarten children possess a high degree of understanding of terms describing premeasurement concepts and are able to recognize several common instruments used in measurement. Approximately 80 per cent or more of the children responded accurately to situations requiring an understanding of *largest*, *smallest*, *tallest*, *longest*, *most*, *inside*, *beside*, *closest*, and *farthest*. A smaller percentage, approximately 50 per cent, recognized situations describing the terms *shortest*, *few*, *underneath*, and *some*.

The response on the oral and written tests pointed out a high degree of ability to recognize common instruments used in measurement. When requested to name the instrument girls and boys would use to perform a specific job, 89 per cent were able to name the clock and 51 per cent could name the calendar. Approximately one-third were able to name accurately the yardstick, foot rule (ruler), scale, and thermometer. Comments worth noting included those by one girl who said, "There are two kinds—weather and sick," when referring to the thermometer, and a boy who suggested his solution for finding out how cold it was by saying, "We could call someone up—the weatherman—or look at the television." Maybe our recent advances in technology have reduced some of our measurement instruments to obsolescence as far as common usage is concerned.

Data secured from the study revealed the ability of about 50 per cent of the five-year-olds tested to recognize time on the full hour when referring to a clock. Fewer possessed the ability to recognize time on the half-hour, as only about one-third of the children responded accurately to situations requiring this knowledge. It would appear that this skill is not beyond the ability of most of these five-year-olds if they are given the opportunity to use the clock in meaningful situations.

There was little verbal response or interest shown when a selection of coins and a dollar bill were displayed on the table. Some commented that it was "a lot of money," another said, "My brother has money," while still another merely stated he had a bank. When asked what they would or could do with the money, little enthusiasm was evoked, with the responses being almost entirely limited to the idea that it was something to play with or to put in one's bank. Eighty per cent of the children immediately recognized the penny, but only 38 per cent recognized a nickel. No particular pattern seemed to develop in the knowledge displayed according to the coins' values. The two extremes were mentioned previously, and they included the coins with the least value. While 39 per cent of the children

knew there were five pennies in a nickel, only 15 per cent knew how many quarters, and but 10 per cent knew how many half dollars there are in a dollar.

It appears that the majority of preschool children possess the ability to recognize the geometric figures of a circle and a square. Ninety-one per cent of the boys and girls tested were able to respond readily and accurately to the illustration of a circle, while 76 per cent recognized the figure of a square. It is interesting to note that it was necessary for the child to possess a knowledge of the names of these two geometric figures in order to respond accurately to their pictured illustrations.

The results of the study seemed to bear out some understanding of simple fractional concepts when about 50 per cent of the children were able to recognize one-half of one item. Eighty-nine per cent recognized an item divided into thirds, and 66 per cent responded accurately to one-fourth of an item. Thirty-three per cent displayed the ability to select one-half of a group of items.

It is interesting to discover the high degree of skill evidenced in solving word problems that involve simple addition and subtraction facts. Almost 90 per cent of the children successfully solved addition combinations, while approximately 75 per cent accurately solved subtraction combinations. The children responded to these problems by marking a series of similar pictures. It appeared to be much more difficult for them to solve more complicated oral problems whose sums or differences exceeded five. To respond correctly, it was necessary for the child to solve the more difficult problems mentally and then indicate the answer on a written number symbol. The test was purposely constructed to contain the more difficult situations, and it was gratifying to note that as high as 35 per cent of the children were able to solve one of the problems successfully. The lowest response was 10 per cent on one problem, but the over-all average of success was 25 per cent of the group being studied.

What are some of the educational implications to be drawn from the responses of these 127 beginning kindergarten children? Certainly some thought should be given to possible adjustments in the presentation of arithmetical concepts in the kindergarten and the first grade. Specifically these might include:

1. A planned arithmetic-readiness program presented at the kindergarten level. This would require a systematic presentation to replace the incidental approach that is now generally employed. It appears that this type of instruction could begin no later than the second semester of the kindergarten year.

2. A program of arithmetic in the first grade that would reflect the readiness period carried out in the kindergarten. The readiness program would be considerably shorter at the first grade level for most children, but could be extended for whatever period necessary for each child before the concepts developed in the usual first grade program are presented. This program would allow more time in the first grade for a broadened experience in arithmetic for most children.

3. An inventory of the child's number concepts should be made during the first part of the kindergarten year so that more emphasis in both the kindergarten and the first grade can be placed on developing number understandings for those children whose background experiences have been more limited than those of the majority. This would appear to be a reasonable approach, inasmuch as the data received from this group of children seem to indicate that the ability of beginning kindergarten children to understand many basic arithmetical concepts is a reality.

EDITOR'S NOTE. In the light of this report, maybe some of our content in grades one and two needs to be examined, particularly with respect to elementary fraction concepts, geometric concepts, basic groupings, and problem solving. Experiences in counting in sequence would seem to be appropriate at an early period. Are coins also becoming obsolete with the younger fry, just as currency is with their credit-card-carrying parents? Seriously, however, we have encountered the response that money is something to put in a piggy bank. Perhaps this is worth a mention at parent conferences if we really want young children to know something about money.

# Comparison of Attitudes and Achievement Among Junior High School Mathematics Classes

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### The Significance of Attitudes in Relation to a Curriculum Study

Attitudes toward arithmetic have long been considered to be of great importance to the educator. It has been generally assumed that those students whose attitudes were more favorable toward the subject achieve at a higher level. It was found in a study of attitudes of junior high school students[1] that most pupils (87%) enjoy problems when they know how to work them well.

In the two junior high schools in Oxnard, California, the mathematics classes at seventh and eighth grade level had been grouped on a basis of arithmetic achievement. This seemed to offer an excellent opportunity to explore further the relationship between the attitudes of the high achievers and those of the low achievers; this study was undertaken to implement this exploration.

### Background of the Problem

During the school year 1958–1959, a curriculum committee composed of teachers, administrators, and county consultants considered the problem of making special provision for seventh and eighth grade students who showed exceptional achievement in mathematics. Contact with other districts indicated that any plans in operation at that time were too new to offer valid information concerning their success. Correspondence with colleges indicated no foreseeable difficulties in presenting algebra at eighth grade level if that plan should be adopted. Efforts to articulate the program with the high school were unsuccessful, however, and the plan adopted did not include algebra.

Final decisions made by the committee, and adopted by the School Board, provided for selection of entering seventh-grade students on the following basis:

1. Standardized test scores
   a. Achievement generally one year above grade level
   b. I.Q. generally 120 or above
2. Teacher opinion
3. Counselor opinion
4. Parents' consent.

The curriculum for those seventh grade students selected was planned to include all the areas of the seventh and eighth grade curriculum made up of new, not review, material. As a guide for material to be presented to these students the following year in eighth grade, a textbook[2] was selected from among many that were examined and evaluated.

[1] Wilbur H. Dutton, "Attitudes of Junior High School Pupils Toward Arithmetic," *School Review*, Vol. 64, January 1956, pp. 18–22.

[2] Francis G. Lankford, Jr. and John R. Clark, *Basic Ideas of Mathematics*, New York: World Book Co., 1953.

One guiding principle of this curriculum change was that continued membership in these classes should be on the basis of ability to move at this faster pace without sacrificing understanding of each new skill. During the first nine weeks of the 1959–1960 school year, seven seventh grade and two eighth grade students were moved into regular classes when it seemed they would benefit more from this type of instruction. In spite of careful counseling in these cases, there was much consternation on the part of some students moved from the accelerated to the regular classes. The hope that a measure of attitudes might give added information which would make selection of students more accurate was also a factor in deciding to carry out this study.

During the school year 1958–1959, in anticipation of the acceptance of this program, a comparable program was presented to the seventh grade pupils, who became the members of the accelerated eighth grade classes of the following school year. The pupils making up these classes were chosen on a basis of previous high achievement and teacher opinion, but not on as accurate a basis as that decided by the curriculum committee.

## Limits of the Problem

This study was limited to the seventh and eighth grade students at one of the two junior high schools in Oxnard School District. There were 610 pupils in the school, 320 of whom were in seventh grade. Pupils included in this study numbered 348.

An attempt was made to answer these questions: (1) Are the attitudes of the group of high achievers significantly different from those of the groups of lower achievers? (2) Will a measure of attitudes toward arithmetic aid in selection of students for accelerated classes?

## Procedures Used

The attitude scale developed by Dutton,[3] as shown in Table I, in the second column, was administered to twelve classes by the writer. Of these classes, six were in the seventh grade and six were in the eighth grade.

TABLE I

SCALE USED IN MEASURING PUPILS' ATTITUDES TOWARD ARITHMETIC

HOW I FEEL ABOUT ARITHMETIC

1. I think about arithmetic problems outside school and like to work them out. (9.5)
2. I don't feel sure of myself in arithmetic. (3.7)
3. I enjoy seeing how rapidly and accurately I can work arithmetic. (8.6)
4. I like arithmetic, but I like other subjects just as well. (5.6)
5. I like arithmetic because it is practical. (7.7)
6. I don't think arithmetic is fun, but I always want to do well in it. (4.6)
7. I am not enthusiastic about arithmetic, but I have no real dislike for it either. (5.3)
8. Arithmetic is as important as any other subject. (5.9)
9. Arithmetic is something you have to do even though it is not enjoyable. (3.3)
10. Sometimes I enjoy the challenge presented by an arithmetic problem. (7.0)
11. I have always been afraid of arithmetic. (2.5)
12. I would like to spend more time in school working arithmetic. (9.0)
13. I detest arithmetic and avoid using it at all times. (1.0)
14. I enjoy doing problems when I know how to work them well. (6.7)
15. I avoid arithmetic because I am not very good with figures. (3.2)
16. Arithmetic thrills me, and I like it better than any other subject. (10.5)
17. I never get tired of working with numbers. (9.8)
18. I am afraid of doing word problems. (2.0)
19. Arithmetic is very interesting. (8.1)
20. I have never liked arithmetic. (1.5)
21. I think arithmetic is the most enjoyable subject I have taken. (10.4)
22. I can't see much value in arithmetic. (3.0)

Within each grade level, two classes were accelerated, two were regular, and two were remedial. The remedial classes are made up of students whose arithmetic achievement scores fall at or below the twentieth percentile.

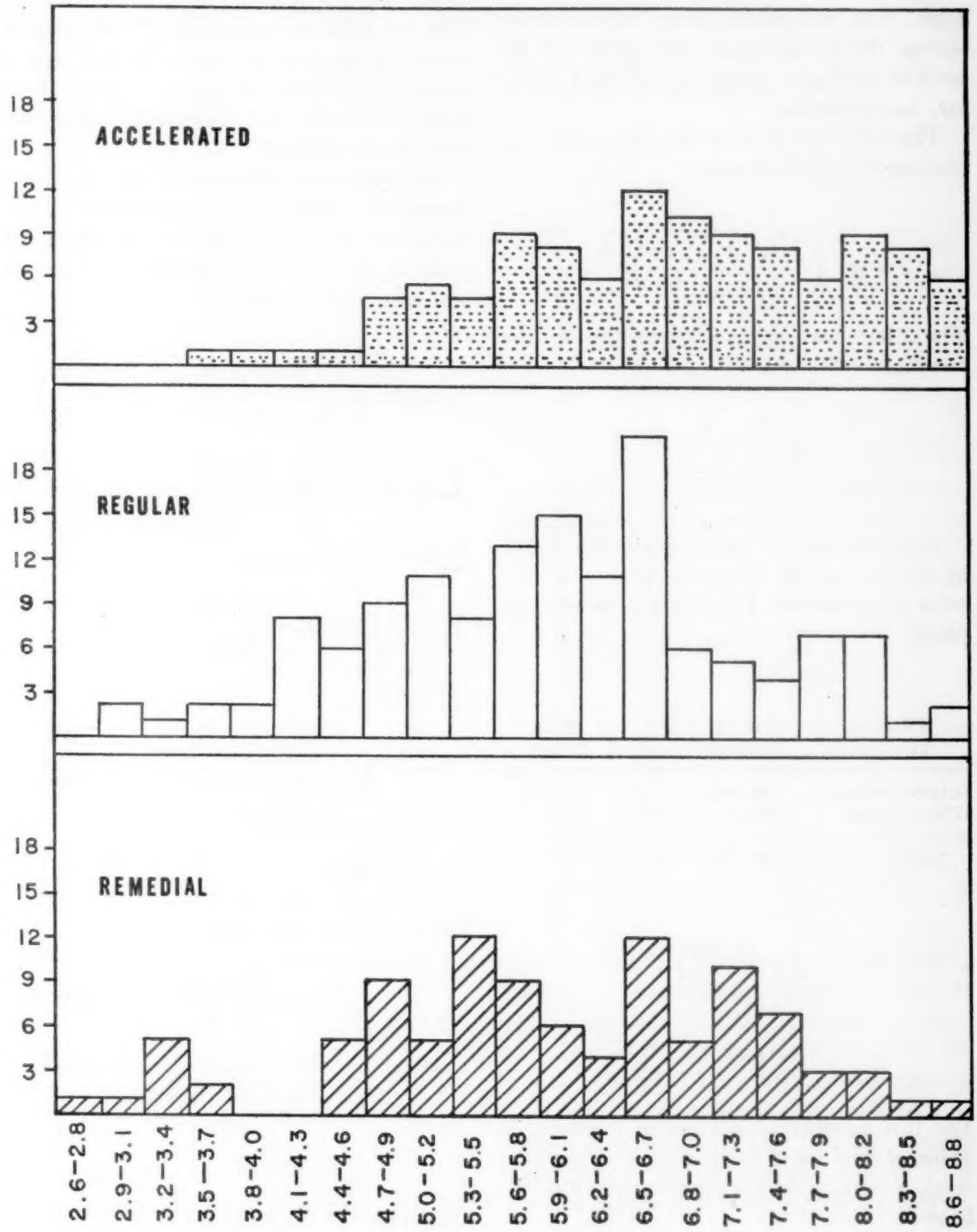
The method of administration used was to discuss with each class the importance of thinking carefully about each statement as it was read to the class. Each pupil was to

[3] Dutton, *op. cit.*, p. 19.

check only those statements that told how he felt most of the time or all the time. It was emphasized that this was not a test and would have no effect on any grade. The students were told they might omit their names on the papers if they cared to, though it was hoped they would identify themselves, and almost all of them did.

TABLE II

HISTOGRAM SHOWING DISTRIBUTION OF SCORES ON ATTITUDE TESTS



After sufficient discussion of the above points, the statements were read to the students, and enough time was allowed after each item for consideration and checking before going on to the next statement.

For each student, the numerical values shown in parentheses after each statement checked were assigned as indicated on the scale in Table I, and the mean score for each pupil was computed. From these mean scores, the group mean was computed for each of the three groups: accelerated, regular, and remedial.

The standard error of the difference was obtained from the formula:

$$s_{\bar{x}_1-\bar{x}_2}=\sqrt{\left(\frac{\sum x_1^2+\sum x_2^2}{n_1+n_2-2}\right)\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$$

The $t$-test was applied according to the formula:

$$t=\frac{\overline{X}_1-\overline{X}_2}{s_{\bar{x}_1-\bar{x}_2}}$$

Since the over-all variance was significant as shown by the following summary, the $t$-test for difference between groups was applied.

| Source of Variation | Sum of Squares | df | Mean Square |
|---|---|---|---|
| Between groups | 50.5953 | 2 | 25.2976 |
| Within groups | 1297.51 | 345 | 3.75 |
| Total | 1348.1053 | 347 | |

$$F=\frac{25.2976}{3.75}$$

$$F=6.75$$

For this number of degrees of freedom, a value of 3.02 for $F$ is significant at the 5% level, a value of 4.66 is significant at the 1% level.

## Analysis of Scores on Attitude Scale

The accelerated group was made up of 108 pupils in the seventh and eighth grades. The range of their scores on the attitude scale was 3.6 to 8.8, with a mean of 6.8. In the regular group, there were 139 pupils from the two grades; the range of their scores was 3.1 to 8.7, with a mean of 6.05. The remedial group contained 101 pupils, whose scores ranged from 2.8 to 8.7, with a mean of 5.95. The histogram in Table II on page 353 shows the distribution of scores after being grouped.

To determine whether the difference between the means of the accelerated and the regular groups was significant, the standard error of the difference between the means was computed as follows:

$$s_{\bar{x}_1-\bar{x}_2}=\sqrt{\left(\frac{\sum x_1^2+\sum x_2^2}{n_1+n_2-2}\right)\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$$

$$\sum x_1^2=5103.21-\frac{543022}{108}$$

$$\sum x_1^2=75.21$$

$$\sum x_2^2=5142.71-\frac{707786}{139}$$

$$\sum x_2^2=50.71$$

$$s_{\bar{x}_1-\bar{x}_2}=\sqrt{\left(\frac{75.21+50.71}{108+139-2}\right)\left(\frac{1}{108}+\frac{1}{139}\right)}$$

$$s_{\bar{x}_1-\bar{x}_2}=\sqrt{.00846363}$$

$$s_{\bar{x}_1-\bar{x}_2}=.0920$$

The $t$-test was then applied:

$$t=\frac{\overline{X}_1-\overline{X}_2}{s_{\bar{x}_1-\bar{x}_2}}$$

$$t=\frac{6.8-6.05}{.0920}$$

$$t=8.1$$

This value of $t$ is significant at the 5% level.

The same computations were carried out to find the difference between the accelerated and the remedial groups with the following results:

$$\sum x_1^2 = 75.21$$

$$\sum x_2^2 = 4747.80 - \frac{361201}{101}$$

$$\sum x_2^2 = 1171.55$$

$$s_{\bar{x}_1-\bar{x}_2} = \sqrt{\left(\frac{75.21+1171.55}{108+101-2}\right)(.00925926+.00990099)}$$

$$s_{\bar{x}_1-\bar{x}_2} = \sqrt{.1153447050}$$

$$s_{\bar{x}_1-\bar{x}_2} = .33962$$

$$t = \frac{6.8-5.95}{.33962}$$

$$t = 2.50$$

This value of $t$ is significant at the 5% level, but not at the 1% level.

Again the same computations were done, this time to find the difference between the regular and the remedial groups. The results follow:

$$s_{\bar{x}_1-\bar{x}_2} = \sqrt{\left(\frac{50.71+1171.55}{139+101-2}\right)(.00719424+.00990099)}$$

$$s_{\bar{x}_1-\bar{x}_2} = .093693$$

$$t = \frac{6.05-5.95}{.093693}$$

$$t = 1.06$$

This value of $t$ is not significant at either the 5% or the 1% level.

## Possible Values Derived from Evaluating Attitudes

CONCLUSIONS

The highly significant difference between the accelerated and the regular groups may indicate that this measure would be helpful in selecting students for future accelerated classes. However, it may be noted in the histogram in Table II that there are in the accelerated group four students who indicate a comparatively negative attitude toward the subject. Because of this, individual interviews may be needed to give insight into the reasons for the less positive feelings held by

students such as these. These interviews might reveal means for improving attitudes, or may suggest that, because of their attitudes, these students might best be placed in a regular class.

The small difference between the accelerated and remedial groups may partially be explained by the fact that these remedial students have also been specially grouped, and this attention to their problem may have improved their attitudes toward the subject. It is also possible that even though the directions were given very carefully, these students may have given less serious consideration to the scale, and may have checked certain items indicating highly positive attitudes because they thought it might secure a measure of attention for them.

The difference between the regular and remedial groups was not a significant one. This would help reinforce the importance of the difference between the accelerated and the average groups, for it is the only highly significant difference found.

RECOMMENDATIONS

It was recommended that this attitude scale be administered to the sixth grade pupils near the end of the school year 1959–1960 to provide one more criterion for selection for membership in the accelerated classes. It appears to the writer that if the student is a doubtful choice on a basis of achievement and ability test scores, and lacks a highly positive attitude toward arithmetic it would be much better not to place that student in the accelerated class. As mentioned, grouping here is flexible. Besides possible changes from an accelerated class to a regular class, there is also the possibility of changing into an accelerated class if, in the opinion of the teacher, the student could do much more work with understanding. If this plan of placement and change is followed, the emotional disturbances observed when reassignments from accelerated to regular classes were made might be avoided, at least to a large degree.

It was further recommended that those sixth grade students showing high ability and achievement scores but exceptionally low scores on the attitude scale be counseled and final placement in seventh grade mathematics classes be made with due consideration to the pupil's attitude toward the placement.

EDITOR'S NOTE. "Nothing succeeds like success" is an old saying. What is cause and what is effect in a situation like this? Mrs. Stephens has, at any rate, given some good information on how to make borderline decisions. This amplifies and justifies a technique used by one of the editors for several years in classifying college students. One of the best ways to gain the necessary information to categorize those college students in beginning mathematics whose entrance test scores are indeterminate is to question them about their success in and liking for grade school arithmetic.

## Boners

"Much of the humor classed as 'schoolboy boners' comes under the head of wrong or inadequate concepts. The boy who thought an average was 'something hens lay eggs on' (he had read the arithmetic problem: Farmer Jones' hens laid 24 eggs on Sunday, 32 eggs on Monday, 28 eggs on Tuesday, etc. How many eggs did they lay on the average during the week?) was suffering from an inaccurate concept."—*Taken from David H. Russell*, Children's Thinking.

# A New Look at the Basic Principles of Multiplication with Whole Numbers

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MUCH HAS BEEN WRITTEN concerning the desirability of stressing the basic principles underlying the fundamental processes. These are usually identified by calling them the Commutative, Associative, and Distributive Laws. Certainly a better understanding of the fundamental processes will result if these principles are understood and stressed by both the teacher and the elementary student. It is the purpose of this article to discuss these principles from the standpoint of their development from a process of logical reasoning, or perhaps we should say from their "reasonableness."

The commutative principle of multiplication asserts that $a \times b = b \times a$. It seems that a certain amount of development of this principle would be necessary if we are to expect the student to have an appreciation for and an understanding of this generalization. Since in arithmetic we are primarily interested in the justification for such a statement, we should test the reasonableness of our conclusions. As an example of the principle $a \times b = b \times a$, we might test such a conclusion as $3 \times 4 = 4 \times 3 = 12$. To do this we draw a configuration, such as,

. . . .

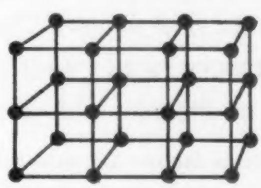
. . . .

. . . .

This figure represents three rows of four dots, or four columns of three dots. At any rate, there are precisely twelve dots in the configuration. We may state in a more abstract terminology that 4 threes = 3 fours = 12.

Obviously, the method by which the pattern was constructed determines to a certain extent the meaning to be assigned to the figure. However, if one does not know how the configuration was developed, each hypothesis is equally likely and desirable, and we may rely upon the commutative principle of multiplication to establish our conclusion that $3 \times 4 = 4 \times 3 = 12$.

Let us now consider the associative principle of multiplication. This states that if three factors are to be considered in finding a product, they may be multiplied in any order. As an example of the application of this principle, let us consider a rectangular solid. It may be thought to be composed of dots arranged to form rows, rows arranged to form layers, and layers piled one upon the other to form a solid, as in the figure below.



Here we may consider that $(3 \times 4) \times 2 = 12 \times 2 = 24$. That is, the total number of dots may be determined by multiplying the number of dots in a row by the number of rows and that product by the number of layers of dots. Since the rows may be organized horizontally, vertically, or from front to back, the layers in a similar manner, we can reason that $(3 \times 4) \times 2 = 3 \times (4 \times 2)$. Since the diagram may be expanded to show any factor size for its dimensions, the reasonableness of the

associative principle for multiplication would seem to be established, and we may assert that

$$a\times(b\times c)=(a\times b)\times c.$$

That is, the same product is obtained regardless of the grouping to perform the multiplications.

Again, a strict interpretation would require a knowledge of the method of construction used to obtain the figure. This type of problem is, of course, the one which is encountered when finding the volume of a solid. Each dot then represents a cubic unit which we can "multiply" in number to obtain the total number of cubic units. The desirability of the associative principle is evident since the product obtained is independent of the mode of construction, and we can reason directly that since $3\times4\times2=24$, there must be 24 cubic units in the figure.

A further investigation of the reasonableness of the associative principle in connection with this problem would show subproducts of $12\times2=3\times8=6\times4=24$. Here we are investigating the interesting observation that three multiplication facts which we have learned obtain for us the same final product, 24. Perhaps we might digress at this point to study further the composition of the product 24. For this purpose we will find the prime factors of 12, 8, 6, and 4 used in obtaining the final product. Thus

$$12=3\times2\times2 \qquad 6=3\times2$$
$$8=2\times2\times2 \qquad 4=2\times2.$$

If these prime factors are substituted for the subproducts which they represent above, we find that

$$12\times2=(3\times2\times2)\times2$$
$$3\times8=3\times(2\times2\times2)=3\times2\times2\times2$$
$$6\times4=(3\times2)\times(2\times2).$$

We have obtained the same set of prime factors in each case. The reasonableness of the equivalence of these multiplication facts thus becomes evident and leads us to consider the fundamental theorem of arithmetic.

## Fundamental Theorem of Arithmetic

This theorem states that for any product there is only one set of prime factors. It asserts that the prime factors of any product can be obtained in various ways, but always with the same final prime factors resulting. The utility of this basic theorem in providing understanding seems to be widely forgotten or neglected in present-day arithmetic instruction. For example,

$$72=12\times6=(6\times2)\times(3\times2)$$
$$=3\times2\times2\times3\times2$$
$$72=9\times8=(3\times3)\times(4\times2)$$
$$=3\times3\times2\times2\times2$$
$$72=18\times4=(9\times2)\times(2\times2)$$
$$=3\times3\times2\times2\times2.$$

Obviously, by rearrangement or commuting, the same order of prime factors can be obtained in each case.

Perhaps we as teachers have neglected the teaching and application of this theorem because we felt that its utility was somehow directly connected only with solving long and difficult problems in the addition of fractions.

Certainly the least common denominators needed for "textbook" problems can usually be obtained by inspection (or perhaps a little intelligent guessing). The power of this theorem to explain products, however, goes far beyond its application to fraction problems. It is by means of this theorem that we can explain many of the product results which we often take for granted.

As an example of the utility of the theorem, we will investigate a simple problem in proportion

$$\frac{24}{36}=\frac{8}{12}\,.$$

We must first note that the two ratios, or fractions, are equal. By cross-multiplying, we see that the two cross-products are equal. This is a test which may be used to determine the equality of any two fractions.

$$24\times12=36\times8=288.$$

A study of the prime factors of these numbers shows more clearly, I believe, the reason why the products are equal.

$$24=2\times2\times2\times3$$
$$12=2\times2\times3$$
$$24\times12=(2\times2\times2\times3)\times(2\times2\times3)$$
$$36=2\times2\times3\times3$$
$$8=2\times2\times2$$
$$36\times8=(2\times2\times3\times3)\times(2\times2\times2)$$

When the prime factors are rearranged, an identity is obtained, and the reason why the cross-products are equal is made more understandable.

We then might conclude that in a proportion, such as

$$\frac{24}{36}=\frac{x}{12},$$

the prime factors of the unknown number, $x$, must be the product of whatever prime factors are in the left-hand cross-product but are not matched by identical prime factors in the right-hand cross-product. Thus $x$ must be in this case $2\times2\times2=8$.

As a further observation, we might note that by rearranging these same prime factors and grouping them differently we can obtain other proportions. Here we begin with the identity

$$288=288.$$

Using two of the possible factorizations for 288, we obtain

$$2\times2\times2\times3\times2\times2\times3$$
$$=2\times2\times3\times3\times2\times2\times2.$$

The factors may be grouped as indicated

$$(2\times2\times2)\times(3\times2\times2\times3)$$
$$=(2\times2)\times(3\times3\times2\times2\times2)$$

or perhaps as

$$(2\times2\times2\times2)\times(3\times3\times2)$$
$$=(2\times2\times3)\times(3\times2\times2\times2).$$

The first grouping might be used to establish the proportion

$$\frac{2\times2\times2}{3\times3\times2\times2\times2}=\frac{2\times2}{2\times3\times3\times2} \text{ or } \frac{8}{72}=\frac{4}{36}.$$

The second grouping might be used to establish the proportion

$$\frac{2\times2\times2\times2}{2\times2\times3}=\frac{3\times2\times2\times2}{3\times3\times2} \text{ or } \frac{16}{12}=\frac{24}{18}.$$

Many variations of the above can be obtained as long as the concept of cross-products being equal is adhered to in each case. Each cross-product must contain precisely five factors of 2 and two factors of 3.

Finally let us consider the distributive principle of multiplication over addition. This principle is of primary importance in explaining the process of multiplication when using a 2-digit multiplier. The reasonableness of the principle can be established with simple numbers by reverse reasoning. Thus, using the multiplication fact $8\times6=8$ sixes, we might reason thus

$$8 \text{ sixes}=6+6+6+6+6+6+6+6.$$

The problem then asks us to find the sum of 8 sixes. To help establish the desired generalization, let us consider two possible groupings of these sixes.

First,

$$(6+6)+(6+6+6+6+6+6)$$
$$2 \text{ sixes} \quad + \quad 6 \text{ sixes}.$$

Next,

$$(6+6+6)+(6+6+6+6+6)$$
$$3 \text{ sixes} \quad + \quad 5 \text{ sixes}.$$

Of course $(2+6)$ sixes $=(3+5)$ sixes $=8$ sixes. In a more abstract sense, we may then think of these groupings as being

$$(2+6)\times6 \text{ or } (3+5)\times6=8\times6.$$

Finally, by the commutative principle of multiplication, they become

$$6\times(2+6) \text{ or } 6\times(3+5)=6\times8.$$

Each of the above now represents an example of the distributive principle.

$$6\times(2+6)=(6\times2)+(6\times6)$$
$$6\times(3+5)=(6\times3)+(6\times5)$$

We now conclude that

$$a\times(b+c)=(a\times b)+(a\times c).$$

We now extend the principle to a more difficult problem, one we cannot solve by use of multiplication facts alone, such as

$$32\times24.$$

Here the 32 must be considered as $30+2$. Then

$$32\times24=(30+2)\times24=(30\times24)+(2\times24)$$
$$=720+48=768.$$

In more conventional notation, this would be

$$\begin{array}{r}24\\ \underline{\times30}\end{array}\quad\text{and}\quad\begin{array}{r}24\\ \underline{\times2}\end{array}.$$

We now obtain both products and add the answers as suggested by the distributive principle.

$$\begin{array}{rl}24&\\ \underline{\times32}&\\ 48&(2\times24)\\ \underline{720}&(30\times24)\\ 768&\end{array}$$

Many more applications of the distributive principle may be found in elementary arithmetic. The principle may be made to apply wherever we consider a factor used to obtain a product as being composed of parts added together. Thus the separate parts may be used to obtain partial products and then these partial products may be added together to obtain the total product desired.

## Summary

The writer has attempted to show that there are only a few basic principles upon which the process of multiplication with whole numbers depends. These principles are called the commutative and associative laws of multiplication and the distributive law of multiplication over addition. Certainly, for completeness, the fundamental theorem of arithmetic should be considered here as well.

Multiplication has been presented as the process of obtaining a "cross-product" from two numbers. This was illustrated by means of a figure in which the number size of the multiplicand was represented by a row of dots and the multiplier was considered as determining the number of rows of these dots needed to make the figure. The writer feels that if students visualize such a pattern as being the result of the use of multiplication as a process, they will use the process more intelligently in problem solving.

The associative principle of multiplication extends the process to situations where three factors are involved. The final product is obtained by multiplying the first two factors together to obtain a product which in turn is multiplied by the third. A final product may be obtained by multiplying the first factor by the product of the next two. An extension of the idea of prime factors to the solution of proportion problems seemed desirable at this point.

While the distributive principle may be used to permit mental or oral computations to be accomplished easily, as in the European schools, it would seem to find its immediate use in terms of explaining multiplication involving a two-digit multiplier. Thus $34\times28$ may be considered as

$$(30+4)\times28$$
$$=(30\times28)+(4\times28)=840+112=952.$$

Of course, the numbers might be further broken down for mental computation as

$$(30+4)\times(20+8)$$
$$=(30\times20)+(30\times8)+(4\times20)+(4\times8)$$
$$=600+240+80+32$$
$$=952.$$

Finally, the fundamental theorem of arithmetic makes a complete analysis of a product possible to determine the simplest "building stones" of which it is composed. These building stones are called prime factors, and they play a vital role in understanding common denominators and in reducing fractions to lower terms.

Since the final goal of the teacher is that of intelligent problem-solving by his student, an understanding of the above principles seems necessary if this purpose is to be accomplished.

EDITOR'S NOTE. There is something for a teacher at almost any level in this article. Lower grade teachers might well elaborate upon the suggestions in the first few paragraphs. Many students who are adept at mental computation will be using variations of these methods. Possibly their explanations of their methods could be used to good advantage as a point of departure for developing Mr. Hannon's examples systematically.

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## Will You Contribute to a Forthcoming Yearbook?

The Board of Directors of The National Council of Teachers of Mathematics at the April, 1960 meeting approved the preparation and publication of a yearbook to be devoted to the problem of the mathematical education of the talented student in grades K-12.

It is intended that the yearbook be a *source-book* of topics and materials which have been found useful in enriching the mathematics program of talented students, but which are not parts of either traditional or experimental courses.

Under the chairmanship of Julius H. Hlavaty, the editorial committee, some members of which are listed below, has accepted the responsibility of preparing the grade-level sections as indicated here. If you have material that you believe would make a contribution toward achieving the purpose of the yearbook, won't you send your contribution to the appropriate member of the committee or to the chairman?

| | |
|---|---|
| K-8 | VINCENT J. GLENNON<br>Director, Arithmetic Center<br>Syracuse University<br>Syracuse 10, New York |
| 7–10 | JOSEPH L. PAYNE<br>3019 University School<br>University of Michigan<br>Ann Arbor, Michigan |
| 9–11 | HENRY SYER<br>Kent School<br>Kent, Connecticut |
| 12, Honors | HARRY D. RUDERMAN<br>Hunter College High School<br>930 Lexington Avenue<br>New York 31, New York |

Please submit your contribution no later than February, 1961. Where grade levels listed here overlap, send the material to either committee member, but not to both. Full credit will be given to each contributor whose material is used.

JULIUS H. HLAVATY
Commission on Mathematics, CEEB
475 Riverside Drive
New York 27, New York

# A Good Teacher—*

*Clarence Ethel Hardgrove, Mildred Cole, and Anne Gustafson*

A GOOD TEACHER of mathematics *learns as he teaches.* This characteristic distinguishes him from a poor teacher. As a result, his teaching becomes better and better as he learns more and more and acts on his knowledge. As he helps boys and girls learn, he, too, is attempting to learn some answers to this problem: "How can I better help children learn mathematics?"

The first step in learning for the teacher is one of *exploration,* of gathering information. This he does by studying children and how they learn, by studying methods of teaching mathematics, and by seeking a deeper insight into the ideas of mathematics.

A good teacher gathers information by studying children and the way they learn mathematics. He obtains information about the intellectual ability, learning rate, achievement level, experience background, and interest of each child, and methods for providing for the differences in these variables which exist among children. He tries methods and observes how the children respond in the learning situation. His answers to questions like these give valuable information: "Was my planning carefully done?" "Were children interested?" "Was each child provided an opportunity to learn?" "Why did learning not result for some?" and, "How will I plan to teach this idea next year?"

A good teacher gains information about the teaching and learning of mathematics by reading about teaching in books, bulletins, and periodicals. There he learns what others consider issues in teaching and what effort is being made to resolve them. There he learns what other teachers are doing in the classroom. He finds many suggestions to add to his store of experiences for developing a particular idea and locates materials for problem-solving and enrichment. He profits from studying reports of research carried out by others.

Information is also gained as a teacher talks with and listens to others who are interested in the teaching and learning of mathematics. Teachers may learn from each other as they share experiences in school hallways, lounges, and teachers' meetings, as they attend conferences and institutes, and as they visit other classrooms. A teacher may learn that a method is poor, that what he does is better than a method used by another teacher, or that a particular idea was successfully taught by a particular method and that the method is worth considering. He may obtain ideas about enrichment or the availability of enrichment material in the community.

A good teacher gains information helpful to his teaching by studying mathematics in order to identify the meanings children are to develop. The greater the insight of the teacher, the better mathematics teacher he is.

The period of exploration, of gathering information about the learning and teaching of mathematics, will not result in improved teaching unless the teacher *thinks* about the information he gathers. All the observing, reading, and listening is of no avail unless he intellectualizes the results of his period of exploration. He must consider, "What does this mean in terms of how children learn?"

*** Excerpt from *Thinking in the Language of Mathematics* (Springfield, Illinois: Illinois Curriculum Program, Superintendent of Public Instruction, 1959).**

and "What does it mean in terms of how I teach?" Thinking of this type results in new ideas about teaching. The teacher may discard some ideas, methods, or materials of instruction; he may keep others in mind for future study; and he may incorporate some into his plans for improved teaching. By means of this procedure, the teacher evaluates methods and materials as he plans for better teaching.

A good teacher then *acts* as a result of his thinking. He has gathered information, considered the problem critically, and arrived at a plan. The action which results may be no change in the method used because the present method seems the best available. It may mean a change in method which results in greater meaning for a mathematical idea. It may mean the use of a different experience so the children will be more interested. It may result in a different classroom organization to provide greater learning opportunity for more of the group. Action is the culminating stage of the learning cycle of a good teacher.

The cycle of exploring, thinking, and acting on a problem is complete, but learning does not stop. Information from this cycle results in greater cause for study, and a new cycle begins. A good teacher is continuously *exploring*, *thinking*, and *acting* as he considers issues of teaching mathematics.

A good teacher is a thinker, a problem-solver. His problem is how best to help children increase their ability to think with the ideas of mathematics.

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## Report of the Nominating Committee

The following named persons have been nominated for the indicated offices in The National Council of Teachers of Mathematics. Biographies and photographs of these nominees will be published in the January issue of THE ARITHMETIC TEACHER.

*Vice President, College Level*

Bruce Meserve, New Jersey
Myron Rosskopf, New York

*Vice President, Junior High School Level*

Helen Garstens, Maryland
Mildred Keiffer, Ohio

*Directors*

Max Beberman, Illinois
Lurnice Begnaud, Louisiana
John A. Brown, Maryland
W. T. Guy, Texas
Julius Hlavaty, New York
Rachel Keniston, California

Ballots will be mailed on or before February 15, 1961 from the Washington office to members of record as of that date. Ballots returned and postmarked not later than March 15, 1961 will be counted.

Respectfully submitted,
OSCAR F. SCHAAF, Chairman

# A Bibliography of Selected Summaries and Critical Discussions of Research on Elementary School Mathematics

Compiled and Edited by J. FRED WEAVER
*Boston University School of Education, Boston, Massachusetts*

SINCE 1957, the last issue of THE ARITHMETIC TEACHER in the spring of each year has included a summary of "Research on Arithmetic Instruction" for the preceding calendar year. (See Reference 37 below.) Your editors plan to continue this service, with the research summary for 1960 appearing in the May 1961 issue.

The bibliography which follows is intended to augment this feature of THE ARITHMETIC TEACHER by listing most of the summaries and critical discussions of research on elementary school mathematics that have been published in one source or another *since* 1940.

The first comprehensive summary of research on arithmetic appeared thirty-five years ago: the *Summary of Educational Investigations Relating to Arithmetic*, Guy T. Buswell and Charles Hubbard Judd (Supplementary Educational Monographs No. 27; University of Chicago Press, 1925). Annual supplements to this summary have appeared in varying forms in *The Elementary School Journal*, compiled by Professor Buswell through Volume 48, and by Prof. Maurice L. Hartung beginning with Volume 49. The annual supplement appeared in the May or June issue through Volume 33, and in the November or December issue beginning with Volume 34. In recent years, these annual listings have been titled, "Selected References on Elementary School Instruction: Arithmetic." Since mention is made of the listings here, they have not been included in the bibliography that follows.

Two recent yearbooks of The National Council of Teachers of Mathematics deserve special mention: *Emerging Practices in Mathematics Education* (Twenty-second Yearbook, 1954) and *Instruction in Arithmetic* (Twenty-fifth Yearbook, 1960). Each has sections of interest to the research worker, although neither yearbook is listed in the following bibliography.

Any reader who has a special interest in a particular research problem will need to look well beyond the listing below. He will need to study at firsthand the reports of research studies that deal specifically with that problem and the helpful summaries of previous related research that are a part of some of these research reports. In any event, the bibliography that follows may serve as a worth-while starting point for his study of research on one phase or another of elementary school mathematics.

1. BEATTY, MRS. LESLIE S. "Re-orienting to the Teaching of Arithmetic." *Childhood Education* 26: 272–278; February 1950.

2. BERNSTEIN, ALLEN. "Library Research—A Study in Remedial Arithmetic." *School Science and Mathematics* 59: 185–195; March 1959.

3. BROWN, KENNETH E. *Analysis of Research in the Teaching of Mathematics—1955 and 1956.* U.S. Dept. of Health, Education, and Welfare; Office of Education; Bulletin 1958, No. 4. Washington, D.C.: U.S. Government Printing Office, 1958. Pp. 16–19, 22–23.

Earlier counterparts of this summary were published in mimeographed form under the following titles:

a. *Mathematics Education Research Studies—1954.* Circular No. 377–III. April 1955.

b. *Mathematics Education Research Studies—1953.* Circular No. 377–II. May 1954.

c. *Mathematics Education Research Studies—1952.* Circular No. 377. July 1953.

4. BROWNELL, WILLIAM A. "Frontiers in Educational Research in Arithmetic." *Journal of Educational Research* 40: 373–380; January 1947.

5. BROWNELL, WILLIAM A. "Teaching of Mathematics in Grades I Through VI." *Review of Educational Research* 15: 276–288; October 1945.

6. BROWNELL, WILLIAM A., and FOSTER E. GROSSNICKLE. "Teaching Mathematics in Grades I Through VI." *Review of Educational Research* 12: 386–404; October 1942.

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8. BURCH, ROBERT L., and HAROLD E. MOSER. "The Teaching of Mathematics in Grades I Through VIII." *Review of Educational Research* 21: 290–304; October 1951.

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(Also preprinted under the same title as pp. 35–45 of the monograph, *Improving the Program in Arithmetic*, published in 1946 by the University of Chicago Press.)

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19. HARTUNG, MAURICE L. "Estimating the Quotient in Division (A Critical Analysis of Research)." THE ARITHMETIC TEACHER 4: 100–111; April 1957.

20. HIGHTOWER, H. W. "Effect of Instructional Procedures on Achievement in Fundamental Operations in Arithmetic." *Educational Administration and Supervision* 40: 336–348; October 1954.

21. HUNNICUTT, C. W., and WILLIAM J. IVERSON (Ed.). "The Third 'R'." *Research in the Three R's.* New York: Harper and Bros., 1958. Pp. 347–429.

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23. JOHNSON, JOHN T. "An Evaluation of Research on Gradation in the Field of

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a. "Research on Arithmetic Instruction: 1958." 6: 121–132; April 1959.

b. "Research on Arithmetic Instruction: 1957." 5: 109–118; April 1958.

c. "Six Years of Research on Arithmetic Instruction: 1951–1956." 4: 89–99; April 1957.

38. WEAVER, J. FRED. "Teacher Education in Arithmetic." *Review of Educational Research* 21: 317–320; October 1951.

39. WEAVER, J. FRED. "Whither Research on Compound Subtraction?" THE ARITHMETIC TEACHER 3: 17–20; February 1956. (Also see Reference 24)

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# In the Classroom

## Groups and Line Arrangements Help Develop Concepts for Numbers in the Span from Ten Through Twenty

Edited by Edwina Deans
*Arlington Public Schools, Arlington, Virginia*

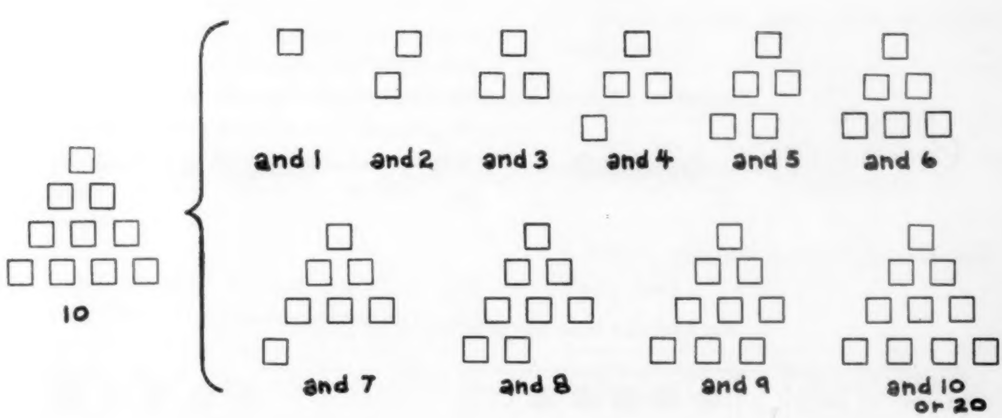
Extensive use of discrete objects was recommended in a previous article to help children develop concepts for the numbers 1 through 9. While there continues to be value in a limited use of discrete objects, you will find it less time consuming to use more structured materials with numbers larger than 10. Counting out 17 objects requires much more time, with more chance for error, than pushing over 10 beads on a frame structured to show 10 and 5 as groups.

Careful teaching is essential to help children gain accurate concepts for the numbers in the span from 10 through 20. The number names 10, 11, and 12 are separate and distinct names for the child in much the same way as the number names 1 through 9 are separate and distinct. With the number 10, however, the number base of 10 begins to operate, giving the "1" used in writing the numeral a group meaning rather than a "single thing" meaning. Now, the one written in tens' place together with the digits 0 through 9 written in ones' place make it possible to represent the numbers 10 through 19.

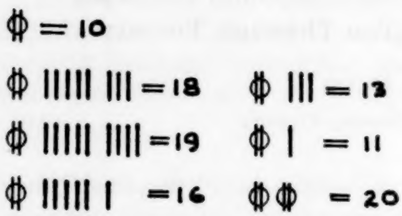
The following activities suggest types of guided experiences that help children develop understanding for numbers in the span from 10 through 20:

1. Arrange 1-inch cubes or squares of cardboard in a triangular pattern to show 10.

   Keeping the group of 10 constant, add one square and read the group; then add two squares and read the group, etc. Arrange the new group beside the 10 to make the 1-, 2-, 3-, 4-patterns which, when complete, will result in another group of 10.



2. Use a bundle of ten sticks and separate sticks to show numbers in the span from 10 through 20. For numbers larger than 15, separate the first 5 ones from the rest to encourage reading the group rather than counting by ones.

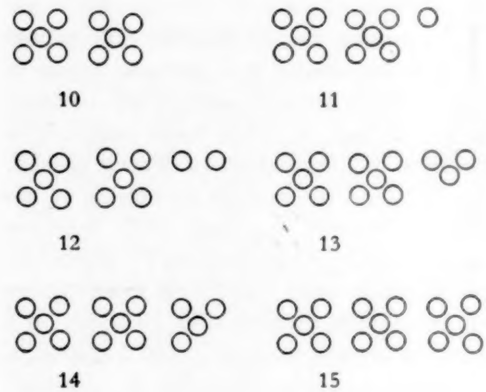



3. Use a 20-bead frame. A satisfactory bead frame can be made from coat-hanger wire on which two colors of plastic or wooden beads are arranged by fives. Leave some free wire to allow for the separation of beads not being used at a given time. Have children reproduce numbers as called, explaining them in terms of 10 and so many more. Use the grouping by fives to help eliminate the need for counting by ones: 15 = 10 and 5; 17 = 10 and 5 and 2.

4. Have children make their own bead lines. A simple bead line may be made from a strip of twenty ½-inch squares of cross-section paper. Children can show numbers called, or show a group named by a numeral written on the chalkboard.

5. Call a number and have children use their bead lines to show the *next* number or the number that comes just *after* the one you call; the number that comes just *before* the number called; the number that is 2 *more* than the number called; the number that is 2 *less* than the number called.

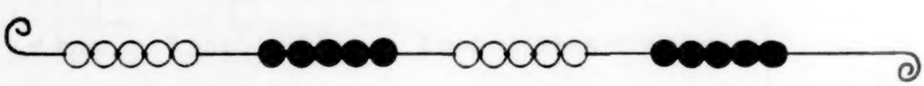
6. Build the numbers 10 through 20 by arranging discs in a 5-pattern:



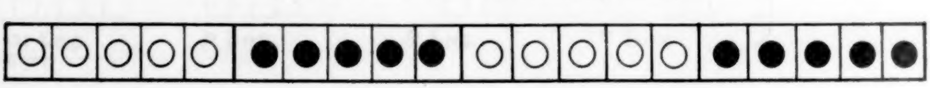

Transfer these groups to picture cards. Have children reproduce the 5-patterns by showing how far each would come on the number bead line. Identify the numbers represented in each case.

7. Help children begin to appreciate the meaning of zero as they learn to write the numeral "10." Explain *zero* as a number meaning "not any."

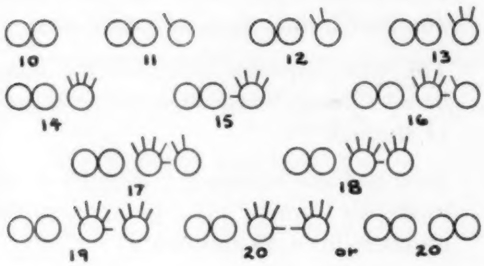
Example of a 20-bead frame for a child to construct



Example of a bead line to be drawn by a child



If children have shown the numbers 1 through 9 with hand pictures, they may now decide on a symbol to stand for all their fingers, or for their two hands clasped together. Let's assume that they decide on two touching ovals for 10. Now they can use this symbol for 10 with hand pictures to show all of the numbers in the span from 10 through 20.




Explain that they must use two places to write numerals for the number of "fingers" in each picture. As children learn to write the numerals from 1 through 20, help them see that each of the numbers from 1 through 9 still has a "how many" meaning. The symbol "1" however, now has a new meaning. It means 1 ten when we write it in tens' place.

8. Give children the opportunity to suggest the make-up of a number as 10 and so many ones: 10 and not any (10); 10 and 1 (11); 10 and 2 (12); 10 and 3 (13). They can read the numbers in another way: *ten, eleven, twelve, thirteen.* With children, stress the fact that the part of the number meaning 10 is written first, although it is heard last as the number names *thirteen* through *nineteen* are spoken.

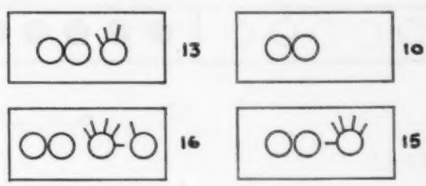
9. Label columns to represent tens and ones. Dictate about five numbers in the span from 10 through 20, having children write the parts of the numbers in their proper places.

| Tens | Ones |
|---|---|
| 1 | 3 |
| 1 | 7 |
| 1 | 2 |

Check immediately for accuracy. Follow by discussion and demonstration to clear up misunderstandings before other numbers are dictated.

10. Call a number between 10 and 20, asking two children to show the meaning of the number. The first child shows the tens' part of the number with clasped hands; the other child shows the ones' part of the number with the appropriate number of outstretched fingers.

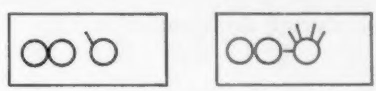
11. Transfer hand number-pictures to cards. Show picture cards and have children write the number each picture represents.



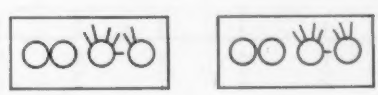

Encourage discussion of why the picture and written numeral belong together.

12. Compare two hand number-pictures at a time.

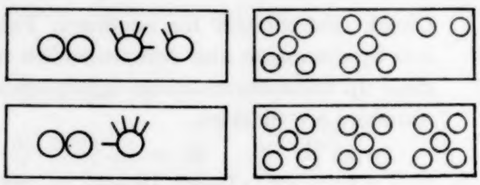
Which picture shows more and how many more?



Which picture shows less and how many less?



13. Show pairs of hand pictures and pictures based on the 5-pattern. Have children determine if pictures represent the same number or two different numbers.



14. For independent work, children may match hand pictures and 5-pattern pictures.

15. Make a number bead line to 20 as suggested, but large enough to be seen from all parts of the room. Tape it to the chalkboard within easy reach of the children. Have them indicate by chalk marks where certain numbers come on the line.

As children engage in activities similar to those described, they begin to develop ideas which are basic to success in mathematics. Among these are the following:

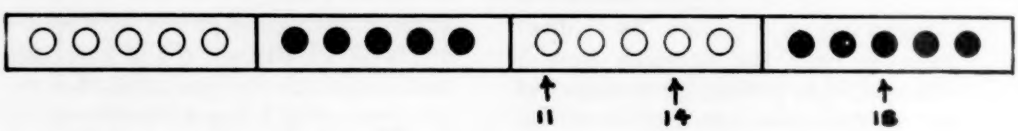
Ten is used as a group.

Zero makes it possible to use each of the numbers 1 through 9 in a new way.

The same sequence noted for the numbers 1 through 9 repeats in the span from 11 through 19.

Zero and the numbers 1 through 9 are used with 1 in tens' place to represent the numbers from 10 through 19.

Example of a number bead line for chalkboard use



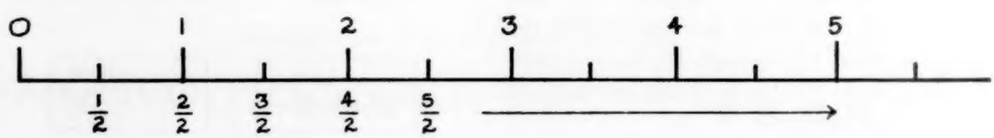

## Using the Number Line to Help Children Understand Fractions

The following excerpts are from a report prepared by Alice Evelyn Shaed for a National Science Foundation sponsored Institute on Teaching Elementary Mathematics held at the University of Michigan, Ann Arbor, during the summer of 1959.

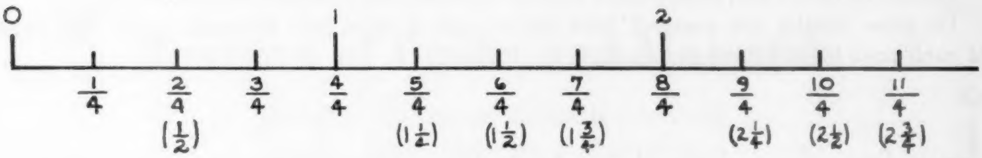
Fractions in the number line help children see that a fraction numeral can serve as a name for a position in a number sequence just as a whole number does. Fraction numerals correspond to points on a number line, just as whole numbers do.

FRACTIONS

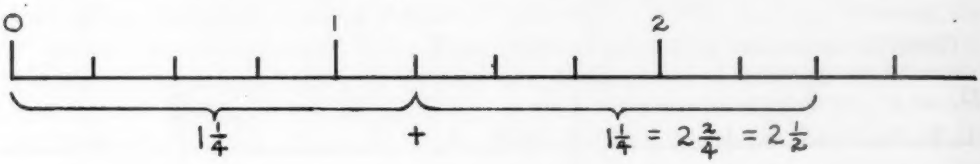
Count by halves:



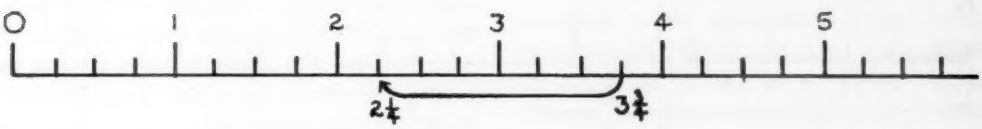
Count by fourths:




Addition of fractions:




Subtraction of fractions:
$3\frac{3}{4} - 1\frac{1}{2} = ?$

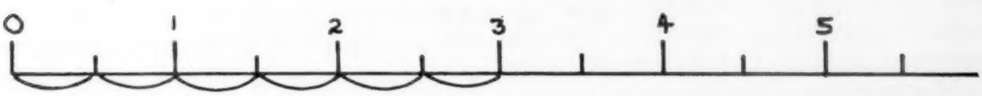



Start at point $3\frac{3}{4}$ and count left $1\frac{1}{2}$ units.

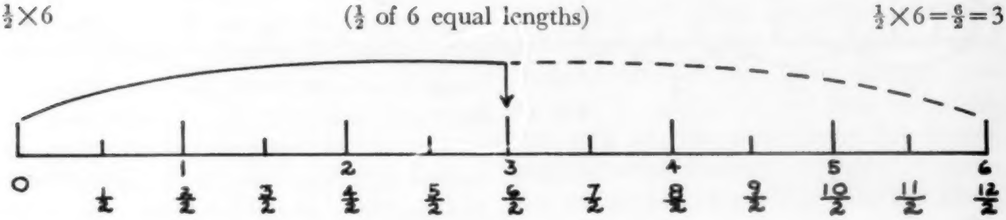
Multiplication of fractions:

(6 steps; each step $\frac{1}{2}$-unit long)

$6 \times \frac{1}{2}$ $6 \times \frac{1}{2} = \frac{6}{2} = 3$

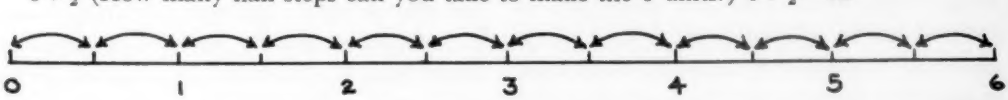



$\frac{1}{2} \times 6$ ($\frac{1}{2}$ of 6 equal lengths) $\frac{1}{2} \times 6 = \frac{6}{2} = 3$




Division of fractions:

$6 \div \frac{1}{2}$ (How many half-steps can you take to make the 6 units?) $6 \div \frac{1}{2} = 12$

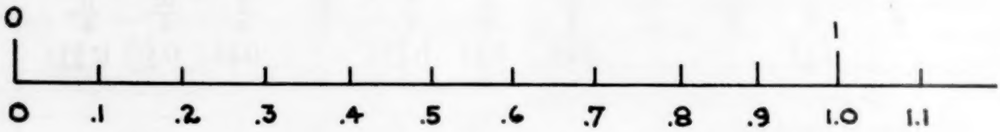



12 half-steps←——3 half-steps, 2 half-steps, 1 half-step

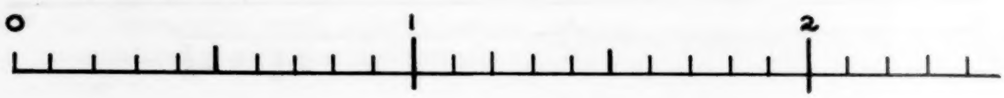
**DECIMAL FRACTIONS**

Tenths, or tenths and hundredths, may be represented with the number line.

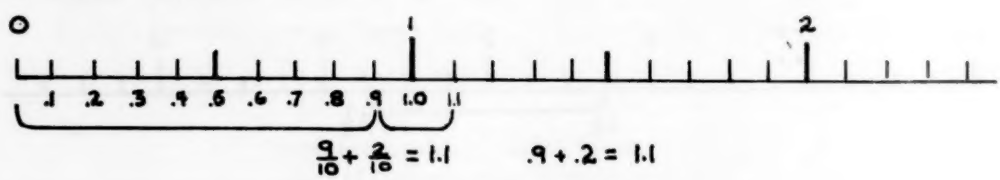
To show tenths, use number lines drawn and divided into 10 equal parts. The value of each part is indicated decimally.



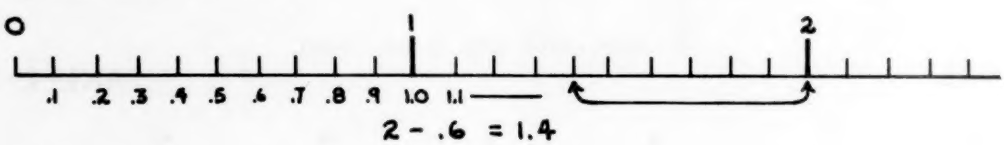

Count by tenths, as: .1, .2, .3, .4 or .2, .4, .6, .8.



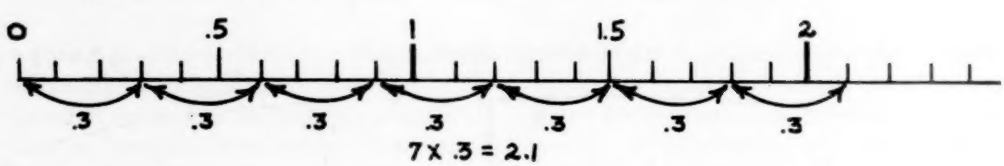

Addition of tenths:



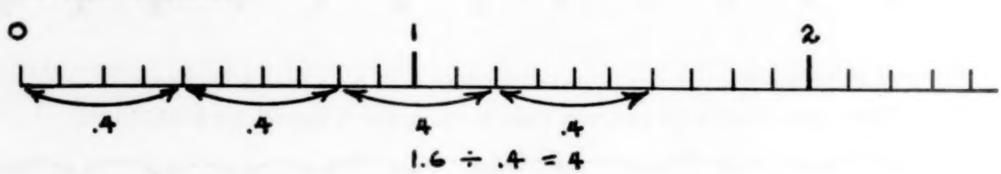

Subtraction of tenths:




Multiplication of tenths:




Division of tenths:



# Reviews of Books and Materials

Edited by Clarence Ethel Hardgrove
*Northern Illinois University, DeKalb, Illinois*

*Instruction in Arithmetic*, Twenty-fifth Yearbook of The National Council of Teachers of Mathematics. Washington, D.C.: The National Council of Teachers of Mathematics, 1960. Cloth, 366 pp. $4.50 ($3.50 to Council members).

The intention of the committee which planned *Instruction in Arithmetic* as a "progress report in clarifying ideas that have already been formulated" is most timely and appropriate. It brings together for ready use by teachers, principals and supervisors, students, and—may the reviewer add—professors of methods classes for teachers of arithmetic, many pertinent developments about the teaching of arithmetic that would take many hours of searching to find in the wealth of literature published on the subject since the appearance of the Sixteenth Yearbook. In this new yearbook, you will not find the organization of topics and chapters so obviously classified as content, methodology, curriculum, evaluation, and the like, as you have learned to expect, although, of course, there are implications throughout on these topics. The committee has, wisely, omitted chapters on evaluation and curriculum (except for primary grades and kindergarten) because future yearbooks have been planned for these areas. It is rather three emphases that guide the organization: (1) the nature of arithmetic, (2) factors such as individual differences, mental hygiene, reading, and guidance, which affect the learning of arithmetic, and (3) the background mathematics recommended for teachers of arithmetic. Since each chapter has been written by separate authors or groups of authors, it seems necessary in reviewing the book to indicate something of the nature of each individual chapter.

Several chapters deal with the nature of arithmetic. In the chapter discussing the cultural value of mathematics, the history of the development of language and number systems and the roles of arithmetic in various past cultures are traced. The meat of the chapter, in the reviewer's opinion, is the pointing out of many interesting uses of arithmetic in various aspects of our culture today, and an indication of the likely role of arithmetic in the automation of the future. A chapter by Van Engen and Gibb shows how physical pattern or structure can be utilized in learning arithmetic to develop the psychical structure which will cause arithmetical situations to fit together and make sense to learners. An intriguing feature of this chapter is the advancement of the idea that rate pairs and fractions, based on structural concepts, have some differences which make it desirable not to think of nor to use per cents as fractions. As an outgrowth of the general acceptance of meaningful arithmetic as the goal of learning activities, methods which encourage children to "discover" and to verbalize generalizations have been advanced. Clark lists several important generalizations in arithmetic, geometry, and business practice and then outlines procedures by which learners may be led inductively to "discover" them. He further suggests series of questions as aids in helping the learner verbalize about these significant generalizations.

In the only chapter dealing directly with curriculum, Spitzer has advanced plans for kindergarten, first, and second grade arithmetic which are generally much broader than those provided in published materials. Many teachers will welcome his proposals as a significant set of experiences for children,

but there will undoubtedly be many who feel that children of the usual ages in these grades are not mature enough to profit from all features of such a program.

Most of the methodology which teachers of arithmetic may study has one overarching aspect of orientation: how best to get the content, as it is traditionally conceived, across. Other concerns of teaching such as individual differences, guidance, reading, and mental hygiene, while recognized, have not had sufficient influence on methodology in arithmetic. Jones and Pingry recognize three groups of causes of individual differences and then suggest means of differentiating the learning activities on the basis of pace, concrete materials, creativity, review, mental computation, estimation, and problem solving. The chapter on "Guidance and Counseling" is a strong appeal for every teacher of arithmetic to be "guidance minded." It describes the procedures by which all the facilities of a school may be used to identify both the slow and the fast learners. It suggests ways to adjust the learning activities and programs to fit these pupils better so that the number with personality and emotional difficulties will be reduced to a minimum. The major concern of the next chapter, "Mental Hygiene and Arithmetic," is "the effect of attitudes upon a successful program in arithmetic." The attitudes of both teachers and pupils receive attention. The authors point out that the newer concepts of content and methodology aid in securing better mental hygiene because everyone concerned operates in an "emotional climate which engenders favorable attitudes toward arithmetic."

The chapter on "Reading in Arithmetic" is divided into two sections. In the first, the nature of reading in arithmetic and some of the peculiarities of the language with which pupils "mathematick" are described. One of the emphases in the second part is the role of reading in solving verbal problems. Twelve suggestions for increasing the learner's "power" in problem solving are presented. Next "Instructional Materials" are classified as exploratory, pictorial, and symbolic. The uses and selection of each kind, including textbooks as symbolic materials, are discussed. Lists of suggested materials, classified for primary, intermediate, or junior high school under subheadings including number and operations, measurement, and relationships are presented, as well as some commercial sources and commercial films and filmstrips.

The impact of the widely publicized appeal to modernize the content of arithmetic is reflected in the next two chapters, the first of which deals with definitions.

After setting up criteria for good definitions and illustrating some violations of them, the authors turn to defining some rather commonly used terms whose definitions have been refined recently by the work in foundations of mathematics. The reviewer has yet to find a better discussion of the terms used in computations involving numbers which are the result of measurements than the one included here in defining such terms as significant digits, possible error, relative error, accuracy, and precision. After introducing the word *set* as an undefined primitive, and discussing cardinal and ordinal numbers in terms of sets, the authors present definitions of disjoint sets, union of two sets, sum of two cardinal numbers, product of two sets, product of two cardinal numbers, fractions, and rational numbers. One of the results pointed out of accepting and using these concepts and definitions in the teaching of arithmetic would be the disappearance of the feeling of need to relate multiplication to addition by insisting on concrete multiplicands and abstract multipliers.

In the second chapter on modernized content, the spirit may be indicated in a brief quotation and a set of three S's. "The child must grasp the place value structure of our numerals before he can understand any standard computational process." The three S's are Set, Scale, and Symbol. The author illustrates how these three S's may be used to engender meaningful learning of arithmetic.

The chapter on "Background Mathematics for Elementary Teachers" is an appeal for a requirement of at least six Semester hours of background mathematics for all elementary teachers, and at least a minor in mathematics for teachers of seventh and eighth grade arithmetic. The content of a background course suggested in this chapter was determined by a questionnaire submitted to the elementary education departments of all members of the AACTE.

The yearbook closes with a rather comprehensive but selective bibliography of pertinent materials published since the Sixteenth Yearbook, classified under sixteen headings. In addition to this bibliography, each chapter, even the introductory one, has its own bibliography, some quite extensive. Each chapter includes a brief review of the main items in its own bibliography.

While it cannot be expected that the publication of a yearbook such as this will revolutionize the teaching of arithmetic overnight, still there is plenty here for all teachers of arithmetic to ponder. Serious evaluation by intelligent teachers of even a few of the suggestions made in this book will no doubt be reflected in better teaching and better success on the part of the learners.

HERBERT F. MILLER

# Christmas Meeting in Tempe, Arizona December 27-30, 1960

## The National Council of Teachers of Mathematics

The Nineteenth Christmas Meeting of The National Council of Teachers of Mathematics will be held in Tempe, Arizona, on December 27–30, 1960, at Arizona State University.

General session speakers will include Dr. Howard Fehr, Dr. T. M. Stinnett, Dr. Leland Carlson, and Mr. J. D. Williams.

Mathematics lectures will be given by Dr. R. A. Dean, Dr. Burton Jones, and Dr. Raymond L. Wilder.

Elementary school sections will feature a fifth grade demonstration class by Professor David Page, the director of the University of Illinois Arithmetic Project and a first grade class demonstration by Professor Patrick Suppes, Stanford University, entitled, "Sets and Numbers: An Approach to Arithmetic via Set Theory."

Participants from the School Mathematics Study Group elementary mathematics writing team will discuss the fourth-grade text and the experimental units for grades five and six. Other talks will deal with the Madison Project, the preparation of special teachers of mathematics, and research in elementary school mathematics.

Junior High School sections will feature the SMSG seventh and eighth grade courses and the texts for the middle group in grades seven and nine. Other talks will deal with the bridge from arithmetic to algebra, problem solving, and the teaching of percentage.

Senior High School sections will review the Developmental Project in Secondary Mathematics at Southern Illinois University, the University of Illinois Project, the Ball State Geometry Program, the coördination of secondary and college mathematics, institutes for mathematics teachers, and a progress report of the SMSG for grades nine to twelve.

The College and Teacher Education sections will hear about a special program at Purdue University, an experimental seminar in science teaching at the University of Arizona, and revised college curricula. The training of teachers in statistics, geometry in the grades, demonstration computers for high schools and colleges, background mathematics for elementary teachers, and the effect of class size on college students' success will also be discussed.

Discussion groups considering the use of NDEA funds in equipping mathematics classrooms will be held at each level. Laboratory sessions at each level will provide a sharing of ideas to implement mathematics teaching.

The Supervision section will discuss the services of the State Supervisor of Mathematics that are available to the schools.

Films and filmstrips will be previewed each day.

# INSTRUCTION IN ARITHMETIC

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